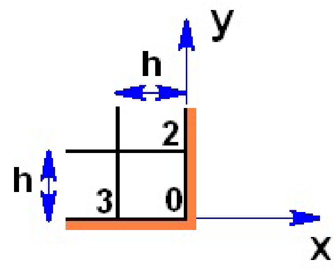
Solution to Problem 134A:

[a] Using the expansions



$$\phi_3 = \phi_0 - h\frac{d\phi}{dx} + \frac{h^2}{2!}\frac{d^2\phi}{dx^2} + O\{h^3\} \tag{1}$$

$$\phi_2 = \phi_0 + h\frac{d\phi}{dy} + \frac{h^2}{2!}\frac{d^2\phi}{dy^2} + O\{h^3\} \tag{2}$$

where all the derivatives refer to the values at the node 0. Since the velocities normal to the wall at the node 0 are zero, it follows that both $\partial\phi/\partial x$ and $\partial\phi/\partial y$ at the node 0 are zero. Then it follows that

$$\frac{d^2\phi}{dx^2} \approx \frac{2(\phi_3 - \phi_0)}{h^2} + O\{h\} \tag{3}$$

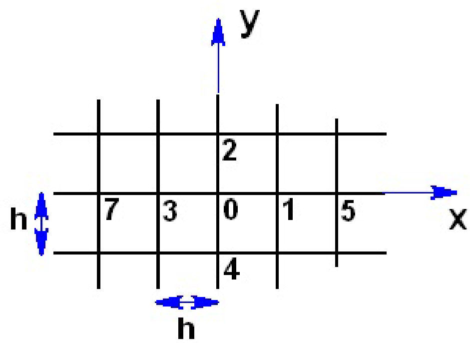
$$\frac{d^2\phi}{dy^2} \approx \frac{2(\phi_2 - \phi_0)}{h^2} + O\{h\} \tag{4}$$

and the Laplace equation for ϕ is approximated by

$$\phi_3 + \phi_2 - 2\phi_0 = 0 \tag{5}$$

where the error is $O\{h^3\}$.

[b] Using the expansions



$$\phi_1 = \phi_0 + h\frac{d\phi}{dx} + \frac{h^2}{2!}\frac{d^2\phi}{dx^2} + \frac{h^3}{3!}\frac{d^3\phi}{dx^3} + \frac{h^4}{4!}\frac{d^4\phi}{dx^4} + O\{h^5\} \tag{6}$$

$$\phi_3 \;=\; \phi_0 - h\frac{d\phi}{dx} + \frac{h^2}{2!}\frac{d^2\phi}{dx^2} - \frac{h^3}{3!}\frac{d^3\phi}{dx^3} + \frac{h^4}{4!}\frac{d^4\phi}{dx^4} + O\{h^5\} \tag{7}$$

$$\phi_5 \;=\; \phi_0 + 2h\frac{d\phi}{dx} + \frac{4h^2}{2!}\frac{d^2\phi}{dx^2} + \frac{8h^3}{3!}\frac{d^3\phi}{dx^3} + \frac{16^4}{4!}\frac{d^4\phi}{dx^4} + O\{h^5\} \tag{8}$$

$$\phi_7 \;=\; \phi_0 - 2h\frac{d\phi}{dx} + \frac{4h^2}{2!}\frac{d^2\phi}{dx^2} - \frac{8h^3}{3!}\frac{d^3\phi}{dx^3} + \frac{16^4}{4!}\frac{d^4\phi}{dx^4} + O\{h^5\} \tag{9}$$

It follows that

$$\frac{d^2\phi}{dx^2} \;=\; \frac{1}{12h^2}\left[16(\phi_1 + \phi_3 - 2\phi_0) - (\phi_5 + \phi_7 - 2\phi_0)\right] + O\{h^5\} \tag{10}$$

[c] The numerical solution, $\phi(x, y)$, is obtained by finding values at each of the nodes that satisfy a numerical version of the Laplace equation, for example:

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0 \;=\; 0 \tag{11}$$

along with the equivalent versions at the boundary nodes. Once that solution has been obtained and the the values at each node have been determined, the velocities $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$ at those nodes can be evaluated using

$$u_0 \;=\; \frac{(\phi_1 - \phi_3)}{2h} \quad ; \quad v_0 \;=\; \frac{(\phi_2 - \phi_4)}{2h} \tag{12}$$

Then, using Bernoulli's equation, the pressure, p_0, at each node 0 (elevation, z_0) may be obtained using

$$\frac{p_0}{\rho} \;=\; \frac{p_\infty}{\rho} \;+\; \frac{U^2 - u_0^2 - v_0^2}{2} \;+\; g(Z - z_0) \tag{13}$$

where p_∞, U and Z are the pressure, velocity and elevation at some reference point in the flow.