Solution to Problem 134B

Obtain the numerical solution to the planar, potential flow around a bend.

a.) Find the velocity potential at all of the nodes and, in particular, the node A.

Any neighboring point can be expanded in a Taylor's Series about a central point, $\phi_{i,j}$.

$$\phi_{i+1,j} = \phi_{i,j} + \left.\frac{\partial \phi}{\partial x}\right|_{i,j} l + \frac{1}{2} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} l^2 + \ldots$$

$$\phi_{i-1,j} = \phi_{i,j} - \left.\frac{\partial \phi}{\partial x}\right|_{i,j} l + \frac{1}{2} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} l^2 + \ldots$$

By adding these equations, we can form an expression for $\frac{\partial^2 \phi}{\partial x^2}$:

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}}{l^2}$$

Similarly the neighboring points in the y-direction can be expanded about the central point to give:

$$\left.\frac{\partial^2 \phi}{\partial y^2}\right|_{i,j} = \frac{\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}}{l^2}$$

Using these two expressions to form Laplace's Equation, we get:

$$\nabla^2 \phi = \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2 \phi}{\partial^2}\right|_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{l^2} = 0$$

$$\Rightarrow \phi_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}}{4}$$

This expression holds at all points in the interior of the grid.

Along a wall, we have a slightly different relationship. At such a point, the velocity normal to the wall is zero. We can no longer write an expansion for $\phi_{i-1,j}$ and we know:

$$u = \left.\frac{\partial \phi}{\partial x}\right|_{i,j} = 0$$

So the expansion for $\phi_{i+1,j}$ becomes:

$$\phi_{i+1,j} = \phi_{i,j} + \frac{1}{2} \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} l^2 + \ldots$$

$$\Rightarrow \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{i,j} = \frac{2\phi_{i+1,j} - 2\phi_{i,j}}{l^2}$$

Again forming Laplace's equation, we see that along such a wall we have:

$$\phi_{i,j} = \frac{2\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i,j-1}}{4}$$

b.) Find the velocity distributions along both the interior and exterior walls.

The velocity midway between two nodes is the difference in ϕ over the cell spacing, l.

$$u_{i,j} = \left.\frac{\partial \phi}{\partial x}\right|_{i,j} = \frac{\phi_{i,j} - \phi_{i+1,j}}{l}$$

$$v_{i,j} = \left.\frac{\partial \phi}{\partial y}\right|_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{l}$$

To find the volume averaged velocity, we sum the flowrates across the inlet and divide by the total area. Each flowrate is given by the velocity midway between two nodes (vertically) multiplied by the cell spacing, l.

$$U = \sum_{j=1}^{5} \frac{q_j}{A} = \frac{\frac{1}{2}\phi_{14,1} + \phi_{14,2} + \phi_{14,3} + \phi_{14,4} + \frac{1}{2}\phi_{14,5}}{4l} = \frac{\bar{Q}}{l}$$

Scaling all of the velocities by the volume averaged velocity:

$$\hat{u}_{i,j} = \frac{u_{i,j}}{U} = \frac{\phi_{i,j} - \phi_{i+1,j}}{\bar{Q}}$$

$$\hat{v}_{i,j} = \frac{v_{i,j}}{U} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\bar{Q}}$$

c.) Find the pressure coefficient, $(p - p_B)/\frac{1}{2}\rho U^2$, along both the interior and exterior walls.

Apply Bernoulli's equation at point B and any other point in the flow.

$$\frac{1}{2}\rho V_B^2 + p_B = \frac{1}{2}\rho V^2 + p$$

$$\Rightarrow \frac{(p - p_B)}{\frac{1}{2}\rho U^2} = \frac{V_B^2}{U^2} - \frac{V^2}{U^2} = \hat{u}_{14.5,5}^2 - (\hat{u}_{i,j}^2 + \hat{v}_{i,j}^2)$$