Solution to Problem 138A

Since by Taylor's expansion:

$$\phi_1 = \phi_0 + h\left(\frac{\partial \phi}{\partial x}\right)_0 + \frac{h^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 + \frac{h^3}{6}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_0 + \frac{h^4}{24}\left(\frac{\partial^4 \phi}{\partial x^4}\right)_0 + \ldots$$

and

$$\phi_3 = \phi_0 - h\left(\frac{\partial \phi}{\partial x}\right)_0 + \frac{h^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 - \frac{h^3}{6}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_0 + \frac{h^4}{24}\left(\frac{\partial^4 \phi}{\partial x^4}\right)_0 - \ldots$$

adding these together we obtain

$$\phi_1 + \phi_3 - 2\phi_0 = 2\frac{h^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 + 2\frac{h^4}{24}\left(\frac{\partial^4 \phi}{\partial x^4}\right)_0 + 2\frac{h^6}{720}\left(\frac{\partial^6 \phi}{\partial x^6}\right)_0 + \ldots$$

Using $2h$ instead of h it also follows that

$$\phi_5 + \phi_7 - 2\phi_0 = 8\frac{h^2}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_0 + 32\frac{h^4}{24}\left(\frac{\partial^4 \phi}{\partial x^4}\right)_0 + 128\frac{h^6}{720}\left(\frac{\partial^6 \phi}{\partial x^6}\right)_0 + \ldots$$

Then eliminating $(\partial^2\phi/\partial x^2)_0$ from the last two equations yields

$$\left(\frac{\partial^4 \phi}{\partial x^4}\right)_0 = \frac{1}{h^4}(\phi_5 + \phi_7 - 4\phi_1 - 4\phi_3 + 6\phi_0) + \quad \text{error of order } h^{-2}$$