Solution to Problem 140B

Part (A)

In its most general form, the equation of continuity can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

which can be expanded as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho = 0$$

Now express this equation in terms of the Lagrangian derivative, D/Dt,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0$$

In this flow, the density of each fluid element is constant. As a result, $D\rho/Dt = 0$. The continuity equation then becomes

$$\nabla \cdot \vec{u} = 0$$

Because each fluid element is incompressible, the momentum equation can be written as

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \vec{F}$$

which can be expanded into the form

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} \right] = -\nabla p + \nabla U$$

where it is assumed the body forces are conservative.

Part (B)

Taking the curl of the momentum equation,

$$\nabla \times \left[\rho \frac{\partial \vec{u}}{\partial t} + \rho\, (\vec{u} \cdot \nabla) \right] = -\nabla \times (\nabla p) + \nabla \times (\nabla U)$$

$$\nabla \times \left(\rho \frac{\partial \vec{u}}{\partial t} \right) + \nabla \times \left[\rho\, (\vec{u} \cdot \nabla)\, \vec{u} \right] = 0$$

$$(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} + \rho \left(\nabla \times \frac{\partial \vec{u}}{\partial t} \right) + \nabla \times \left[\rho\, (\vec{u} \cdot \nabla)\, \vec{u} \right] = 0$$

Evaluating this expression at time $t = 0$ when $u = 0$ but $\partial u/\partial t \neq 0$ it follows that

$$\left[(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} \right]_{t=0} = \left(-\rho \frac{\partial \vec{\omega}}{\partial t} \right)_{t=0}$$

Thus

$$\left(\frac{\partial \vec{\omega}}{\partial t} \right)_{t=0} = -\frac{1}{\rho} \left[(\nabla \rho) \times \frac{\partial \vec{u}}{\partial t} \right]_{t=0}$$