Solution to Problem 140C

Part (A)

The velocity profile for Couette flow is linear:

$$u(y) = \frac{U}{h} y$$

where U is the velocity of the moving plate, h is the distance between the two plates, and y is measured in a direction normal to the plates. The vorticity is defined as

$$\vec{\omega} = \nabla \times \vec{u}$$

and its magnitude in planar flow is therefore given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In Couette flow, $\partial v / \partial x = 0$, and the (magnitude of the) vorticity is

$$\omega = -\frac{\partial u}{\partial y} = -\frac{U}{h}$$

Part (B)

In planar Poisseuille flow, the velocity profile is

$$u(y) = \frac{1}{\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{y}{2}(H-y) = \frac{1}{\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{H}{2}y - \frac{y^2}{2}\right)$$

and therefore the vorticity is given by

$$\omega = -\frac{\partial u}{\partial y}$$

which becomes

$$\omega = -\frac{1}{\mu}\left(-\frac{\partial p}{\partial x}\right)\left(\frac{H}{2} - y\right)$$

Part (C)

The Couette flow of problem 150D had a velocity profile given as

$$u(y) = \frac{8Uy}{5H} - \frac{3U}{5} \qquad \text{for } y > \frac{H}{2} \qquad \text{and} \qquad = \frac{2Uy}{5H} \qquad \text{for } y < \frac{H}{2}$$

The vorticity is

$$\omega = -\frac{\partial u}{\partial y}$$

which becomes

$$\omega = -\frac{8U}{5H} \qquad \text{for } y > \frac{H}{2} \qquad \text{and} \qquad = -\frac{2U}{5H} \qquad \text{for } y < \frac{H}{2}$$

Part (D)

The velocity profile for the flow in problem 150B is

$$u(y,t) = U^* e^{kt} e^{-\sqrt{\frac{\rho}{\mu}k}y}$$

Therefore the vorticity is

$$\omega = -\frac{\partial u}{\partial y}$$

which becomes

$$\omega = \sqrt{\frac{\rho}{\mu}k} U^* e^{kt} e^{-\sqrt{\frac{\rho}{\mu}k}y}$$

Part (E)

The velocity profile for steady, vortical flow is given as

$$u_\theta(r) = Ar + \frac{B}{r}$$

where A and B are constants. The definition of the vorticity is

$$\vec{\omega} = \nabla \times \vec{u} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta}\right) \times \left[\hat{\theta}\left(Ar + \frac{B}{r}\right)\right] = \hat{z}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[\left(Ar + \frac{B}{r}\right)r\right]\right\} = \hat{z}\left[\frac{1}{r}(2Ar)\right] = \hat{z}\,(2A)$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ are unit vectors in the r-, θ-, and z-directions, respectively. Thus, the magnitude of the vorticity vector is

$$\omega = 2A$$