Solution to Problem 147A

1.) First determine the kinematic boundary condition on the components of the velocities normal to the interface in the liquid and vapor phases, u_{Ln} and u_{Vn}. Since any mass transfer across the interface must conserve mass, the mass fluxes on either side must match:

$$\dot{m}_L = \dot{m}_V$$

$$\rho_L u_{Ln} = \rho_V u_{vn}$$

2.) Find the kinematic boundary condition on the components of the velocities tangential to the interface in the liquid and vapor phases, u_{Ls} and u_{Vs}. Just as at a solid boundary, the no-slip condition must apply at the interface and the tangential velocities must match:

$$u_{Ls} = u_{Vs}$$

3.) Find the dynamic boundary condition on the shear stresses in the liquid and vapor phases, σ_{Lsn} and σ_{Vsn}. Consider the momentum theorem in the s-direction:

$$\sum F_s = \int \rho u_s (\vec{u} \cdot \vec{n}) dA$$

and apply this to an infinitesimally thin control volume of area, A, on the interface. The thinness of the control volume means that we can neglect all forces and momentum fluxes through the side walls of the CV. Keep in mind that there will be tangential momentum fluxes that cross the two non-infinitesmal sides of the control volume.

$$\sigma_{Vsn} A - \sigma_{Lsn} A = \rho_V u_{Vs} u_{Vn} A - \rho_L u_{Ls} u_{Ln} A$$

But since $\rho_L u_{Ln} = \rho_V u_{Vn}$ from part (1) and $u_{Ls} = u_{Vs}$ from part (2), the RHS is equal to zero so the shear stresses across the interface must match:

$$\sigma_{Lsn} = \sigma_{Vsn}$$

$$\mu_L \frac{\partial u_{Ls}}{\partial n} = \mu_V \frac{\partial u_{Vs}}{\partial n}$$

4.) Find the dynamic boundary condition on the normal stresses in the liquid and vapor phases, σ_{Lnn} and σ_{Vnn}. For the normal stresses, we consider the momentum theorem applied in the normal direction using the same control volume used in part (3).

$$\sum F_n = \int \rho u_n (\vec{u} \cdot \vec{n}) dA$$

$$\sigma_{Vnn} A - \sigma_{Lnn} A = -\rho_L u_{Ln}^2 A + \rho_V u_{Vn}^2 A$$

$$\Rightarrow \sigma_{Vnn} - \sigma_{Lnn} = \rho_V u_{Vn}^2 - \rho_L u_{Ln}^2$$

Aside: How much mass transfer (evaporation/condensation/diffusion) will have to occur to significantly affect the pressure difference across the interface?

The pressure (the negative of the normal stress, $p = -\sigma_{nn}$) difference across the interface is given by

$$\begin{aligned}
p_L - p_V &= \rho_V u_{Vn}^2 - \rho_L u_{Ln}^2 \\
&= \rho_V u_{Vn}^2 - \frac{\rho_V^2}{\rho_L} u_{Vn}^2 \\
&= \rho_V u_{Vn}^2 \left(1 - \frac{\rho_V}{\rho_L}\right) \\
&\approx \rho_V u_{Vn}^2
\end{aligned}$$

The second step uses the result from part (1). The last step makes use of the fact that the density of a vapor is usually much less than that of a liquid, $\frac{\rho_V}{\rho_L} \ll 1$. It follows that

$$\begin{aligned} \frac{p_L}{p_V} - 1 &= \frac{\rho_V u_{Vn}^2}{p_V} \\ &= \frac{\rho_V u_{Vn}^2}{\rho_V R T_V} \\ &\propto M_V^2 \end{aligned}$$

where M_V is the Mach number of the normal flow in the vapor. The second step used the ideal gas law and the third utilized the fact that the speed of sound in the vapor is proportional to the square-root of RT_V. This demonstrates that for any substantial pressure difference across the interface, the Mach number of the normal vapor flow must be some significant fraction of one. A Mach number such as this would require an intense level of evaporation or condensation.