Solution to Problem 160A

The four given characteristics of this flow are

1. The flow is steady: $\frac{\partial}{\partial t} \equiv 0$
2. There is no swirl: $u_\theta = 0$, $\frac{\partial}{\partial \theta} = 0$
3. The flow is fully-developed: $\frac{\partial u_z}{\partial z} = 0$
4. No body forces are present: $f_r = f_\theta = f_z = 0$

The continuity equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Applying the above characteristics, this becomes

$$\frac{\partial}{\partial r}(ru_r) = 0$$

from which it follows that

$$ru_r = \text{constant}$$

Given that $u_r = 0$ at $r = R$, the constant on the right hand side must equal zero and consequently u_r must equal zero everywhere in order for this expression to hold for all r. Thus u_z is the only non-zero velocity component.

The Navier-Stokes equations in the r and θ directions are

$$\rho\left(\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + f_r + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\rho\left(\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

where the operators D/Dt and ∇^2 are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

After applying the characteristics, the Navier-Stokes equations become

$$\frac{\partial p}{\partial r} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial \theta} = 0$$

which means the pressure is a function only of z, $p = p(z)$. From the Navier-Stokes equation in the z-direction,

$$\rho\frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + f_z + \mu\nabla^2 u_z$$

which after applying the above simplifications yields

$$0 = -\frac{dp}{dz} + \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right)$$

Rearranging

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} = \frac{1}{\mu}\frac{dp}{dz}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{1}{\mu}\frac{dp}{dz}$$

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2}\frac{1}{\mu}\frac{dp}{dz} + A$$

and, after integrating with respect to r, this yields

$$u_z = \frac{r^2}{4\mu}\frac{dp}{dz} + A\ln r + B$$

Since u_z must be finite at $r = 0$, it follows that $A = 0$. In addition, the no-slip condition requires that $u_z = 0$ at $r = R$ and this determines B. Then

$$u_z = \frac{1}{4\mu}\left(-\frac{dp}{dz}\right)\left(R^2 - r^2\right)$$

In this problem, the pressure gradient dp/dz is given:

$$\frac{dp}{dz} = -\frac{\rho g H}{L}$$

which is the pressure difference resulting from the difference in water levels between the two tanks divided by the length of the pipe. Thus

$$u_z = \frac{1}{4\mu}\frac{\rho g H}{L}\left(R^2 - r^2\right)$$

Then the mass flow rate, Q, is obtained by integration:

$$Q = \int_0^R u_z\,(2\pi r)\,dr$$

$$Q = \int_0^R \frac{\pi}{2\mu}\frac{\rho g H}{L}\left(R^2 r - r^3\right)dr$$

$$Q = \frac{\pi\rho g H}{2\mu L}\left[\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=R}$$

and therefore

$$Q = \frac{\pi R^4 \rho g H}{8\mu L}$$