Solution to Problem 160C:

The equilibrium equations (the equations of motion in terms of the stresses) yield (for $y > 0$):

$$\rho \frac{Du}{Dt} \;=\; \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \;=\; -\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\left\{ c \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} \tag{1}$$

$$\rho \frac{Dv}{Dt} \;=\; \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \;=\; -\frac{\partial p}{\partial y} - \frac{\partial}{\partial x}\left\{ c \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} \tag{2}$$

Since the flow is planar and fully developed $v = 0$ and $\partial u/\partial x = 0$ and it follows from the second equation above that $\partial p/\partial y = 0$. Therefore p is a function only of x and the quantity $-dp/dx$ may be regarded as the imposed pressure gradient. It also follows from the first equation that

$$\frac{\partial}{\partial y}\left\{ \left(\frac{\partial u}{\partial y} \right)^2 \right\} \;=\; -\frac{1}{c}\frac{\partial p}{\partial x} \;=\; \frac{1}{c}\left(-\frac{dp}{dx} \right) \tag{3}$$

and since u is a function only of y we may integrate this relation and use the boundary conditions (1) that $\partial u/\partial y = 0$ on $y = 0$ and (2) that $u = 0$ on $y = h/2$ to obtain

$$u \;=\; \frac{2}{3}\left\{ \left(\frac{h}{2} \right)^{3/2} - y^{3/2} \right\} \left\{ \frac{1}{c}\left(-\frac{dp}{dx} \right) \right\}^{1/2} \tag{4}$$

The mean velocity, $\overline{u}$, is

$$\overline{u} \;=\; \frac{2}{h}\int_0^{h/2} u\, dy \;=\; \frac{2}{5}\left(\frac{h}{2} \right)^{3/2} \left\{ \frac{1}{c}\left(-\frac{dp}{dx} \right) \right\}^{1/2} \tag{5}$$