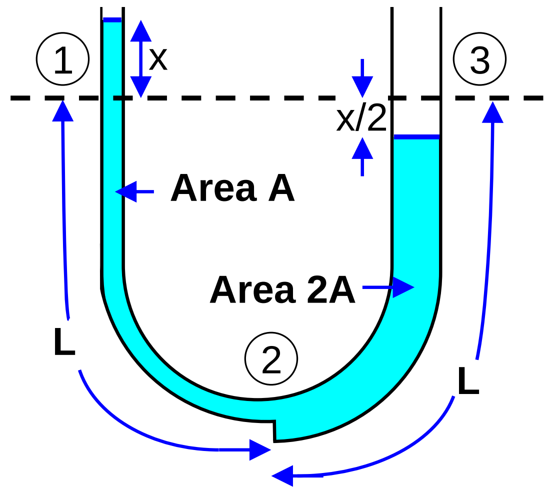
Solution to Problem 206A

The U-tube shown in the figure has one side of length, L, and cross-sectional area, A, and the other side with the same length but a cross-sectional area, $2A$:



Assume no friction within the pipe and an incompressible fluid. During the oscillation, assume the level of the fluid in the left hand side of the tube (point (1)) rises a distance $y_1 = x$. Because of volume conservation, the level at the right-hand side (point (3)) will drop to a level $y_2 = -x/2$. The velocity and acceleration on the left hand side (denoted as positive in the direction from point (3) to point (1)) are $u_1 = dx/dt$ and $a_1 = d^2x/dt^2$ while these quantities on the right hand side are $u_2 = 0.5\ dx/dt$ and $a_2 = 0.5\ d^2x/dt^2$.

Denote the point where the area changes abruptly as point (2). Then the total pressure difference $P_1 - P_3$ can be determined by applying the unsteady Bernoulli equation twice between point (1) and point (2) and between point (3) and point (2):

$$P_2 - P_1 = \rho L \frac{du_1}{dt} = \rho L \frac{d^2x}{dt^2}$$

and

$$P_3 - P_2 = \rho L \frac{du_2}{dt} = \frac{1}{2} \rho L \frac{d^2x}{dt^2}$$

Adding the two equations gives:

$$P_3 - P_1 = \frac{3}{2} \rho L \frac{d^2x}{dt^2} \tag{1}$$

But also by definition the difference between the total pressures at points (1) and (2) is

$$P_1 - P_3 = \left(p_a + \frac{1}{2}\rho u_1^2 + \rho g y_1 \right) - \left(p_a + \frac{1}{2}\rho u_3^2 + \rho g y_3 \right)$$

For small amplitudes of motion the kinetic energy terms involving $0.5\rho u_1^2$ and $0.5\rho u_3^2$ are negligible and since $y_1 = x$ and $y_3 = -x/2$ it follows that:

$$P_1 - P_3 = \frac{3}{2} \rho g x \tag{2}$$

Hence

$$\frac{d^2x}{dt^2} + \frac{g}{L} x = 0$$

which is similar to a pendulum. The natural frequency of the oscillation inside the tube is therefore:

$$\omega = \sqrt{\frac{g}{L}}$$