Solution to Problem 210A:

The flow in the pipeline will be governed by the unsteady Bernoulli equation so that

$$g\,\Delta H \;=\; \rho\frac{l}{A}\frac{dQ}{dt}+\frac{\rho k}{2}\left(\frac{Q}{A}\right)^2 \tag{1}$$

where ΔH is the head rise across the pump and the head loss in the pipeline, Q is the volume flow rate through the pipeline, A is the cross-sectional area of the pipeline, k is the loss coefficient for the pipeline, ρ is the fluid density, t is time and g is the acceleration due to gravity. With $k = fL/D$ (f is the friction factor and L and D are the length and diameter of the pipe) this can be written as

$$\Delta H \;=\; C_1\frac{dQ}{dt}+C_2\,Q^2 \tag{2}$$

where

$$C_1 \;=\; \frac{L}{gA} \;=\; 24336\ s^2/m^2 \quad\text{and}\quad C_2 \;=\; \frac{fL}{2gDA^2} \;=\; 4841\ s^2/m^5 \tag{3}$$

Question (i): At $Q = 0$, $\Delta H = C_1\ dQ/dt$ therefore

- Pump(a)

$$\frac{dQ}{dt} \;=\; \frac{\Delta H}{C_1} \;=\; \frac{200}{24336} \;=\; 0.0082\ m^3/s^2 \tag{4}$$

- Pump(b)

$$\frac{dQ}{dt} \;=\; \frac{\Delta H}{C_1} \;=\; \frac{(200-1000Q}{24336} \;=\; 0.0082\ m^3/s^2\ \text{since Q=0} \tag{5}$$

Question (ii): The asymptotic flow rate, $Q(\infty)$:

- Pump(a)

$$Q(\infty) \;=\; \left(\frac{\Delta H}{C_2}\right)^{1/2} \;=\; \left(\frac{200}{4841}\right)^{0.5} \;=\; 0.203\ m^3/s \tag{6}$$

- Pump(b): Need to solve the quadratic equation

$$\Delta H \;=\; 200-1000Q \;=\; C_2\,Q^2 \tag{7}$$

  which yields $Q(\infty) = 0.125\ m^3/s$

Question (iii): To find $Q(t)$: The following equation applies:

$$\Delta H \;=\; C_1\frac{dQ}{dt}+C_2\,Q^2 \tag{8}$$

and therefore the integral that must be evaluated to determine $Q(t)$ is

$$t \;=\; \int_0^Q \frac{C_1}{\Delta H - C_2 q^2}dq \tag{9}$$

where q is a dummy variable.

- Pump(a): For pump(a) this becomes

$$t = \int_0^Q \frac{C_1}{200 - C_2 q^2} dq \qquad (10)$$

- Pump(a): For pump(b) this becomes

$$t = \int_0^Q \frac{C_1}{200 - 1000q - C_2 q^2} dq \qquad (11)$$

These may require numerical integration.