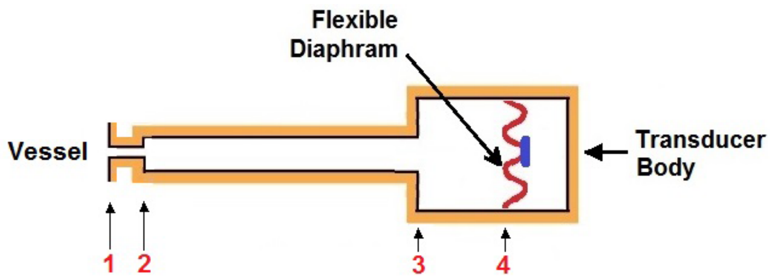
Solution to Problem 210C:

The above diagram represents the connecting components of the transducer:



- Section $1-2$, the pressure tap has a diameter of $1mm$, a cross-sectional area, A_1 and a length of $L_1 = 2cm$.
- Section $2-3$, the connection tube has a diameter of $5mm$, a cross-sectional area, A_2 and a length of $L_2 = 30cm$.
- Section $3-4$, the chamber of the transducer has a diaphragm area of $1cm^2 = 10^{-4}m^2$ and a deflection of $0.1mm = 10^{-4}m$ for each $1atm$ of pressure change. Therefore the compliance, C, of the transducer containing water is defined by $C = \rho dV/dp$ ($\rho \approx 1000kg/m^3$ is the density of the water in the transducer, the piping and the vessel) and is

$$C = \frac{(1000kg/m^3)(10^{-4}m^2)(10^{-4}m)}{(101325kg/ms^2)} = 9.87 \times 10^{-11}\ ms^2 \tag{1}$$

The equations governing the dynamics of this system are the unsteady Bernoulli equation:

$$\rho\frac{\partial\phi}{\partial t} + \rho gH = \rho\frac{dQ}{dt}\int\frac{1}{A}dx + \rho gH = \text{uniform} \tag{2}$$

and the continuity equation, dQ/dt is uniform, independent of location. Applying this to Section $1-2$:

$$\rho\frac{dQ}{dt}\int_1^2\frac{1}{A}dx + \rho g(H_2 - H_1) = \rho\frac{dQ}{dt}\frac{L_1}{A_1} + \rho g(H_2 - H_1) = 0 \tag{3}$$

Applying this to Section $2-3$:

$$\rho\frac{dQ}{dt}\int_2^3\frac{1}{A}dx + \rho g(H_3 - H_2) = \rho\frac{dQ}{dt}\frac{L_2}{A_2} + \rho g(H_3 - H_2) = 0 \tag{4}$$

Combining equations (3) and (4):

$$\rho\frac{dQ}{dt}\left(\frac{L_1}{A_1} + \frac{L_2}{A_2}\right) + \rho g(H_3 - H_1) = 0 \tag{5}$$

If we represent the oscillating components of Q, H_1 and H_3 by

$$Q = Re\left\{\tilde{Q}e^{j\omega t}\right\} \ ; \ H_1 = Re\left\{\tilde{H}_1 e^{j\omega t}\right\} \ ; \ H_3 = Re\left\{\tilde{H}_3 e^{j\omega t}\right\} \tag{6}$$

where ω is the radian frequency of the oscillations then equation (5) yields

$$j\omega\rho\tilde{Q}\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}\right)+\rho g(\tilde{H}_3-\tilde{H}_1) \;=\; 0 \tag{7}$$

In the transducer chamber, the volume of water, V, is related to Q by

$$Q \;=\; \frac{dV}{dt} \;=\; \frac{dV}{dp}\frac{dp}{dt} \;=\; \frac{C}{\rho}\frac{d(\rho g H_3)}{dt} \;=\; Cg\frac{dH_3}{dt} \tag{8}$$

and therefore the oscillations are governed by

$$\tilde{Q} \;=\; j\omega C g \tilde{H}_3 \tag{9}$$

and combining equations (7) and (9):

$$\tilde{H}_1 \;=\; \tilde{H}_3\left[1-\omega^2 C\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}\right)\right] \tag{10}$$

The natural frequency pertains when $\tilde{H}_3$ can be non-zero even when $\tilde{H}_1$ is zero or very small and the above yields a natural frequency of

$$\left[C\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}\right)\right]^{-1/2} \tag{11}$$

Given the above values the natural frequency is $499 rad/s$ or $79.6 Hz$.

The effect of a bubble: a small bubble in the transducer chamber will change the compliance of the fluid in that chamber. The compliance, C_b, of the bubble is given by $\rho V_g/\gamma p$ where V_g is the volume of the bubble, p is the pressure and $\gamma = 1.4$. Therefore $C_b = 9.97 \times 10^{-11} m/s^2$ and the new compliance of the fluid in the transducer chamber is the sum of C and C_b. This yields a modified natural frequency of $352 rad/s$ or $56 Hz$.