Solution to Problem 220A

The continuity equation in integral form can be written as

$$\int_S \rho \vec{u} \cdot \vec{n} dA = 0,$$

where S is the surface of the control volume, $\vec{n}$ is an outward-pointing unit vector normal to the surface, and dA is an elemental surface area. In this case, the integral leads to the relation

$$\rho V h_1 = \rho V_2 h_2.$$

or

$$V_2 = V \frac{h_1}{h_2}.$$

The x-momentum equation in integral form can be written as

$$\sum F_x = \int_S u \rho \vec{u} \cdot \vec{n} dA$$

where $\sum F_x$ is the total force in the x-direction on the control volume. In this case, it leads to

$$\sum F_x = V \rho(-V) h_1 b + V_2 \rho(V_2) h_2 b$$

where b is the width of the gate in the direction normal to the sketch. Two forces act on the control volume: (1) the net force due to the pressures acting on the ends of the control volume, F_p, and (2) the reaction force necessary to hold the gate in place, F_r. Thus, the x-momentum equation becomes

$$F_p - F_r = \rho V_2^2 h_2 b - \rho V^2 h_1 b$$

where

$$F_p = F_{p1} - F_{p2} = \int_0^{h_1} \rho g b y dy - \int_0^{h_2} \rho g b y dy = \frac{1}{2} \rho g b \left(h_1^2 - h_2^2 \right)$$

Substituting

$$\frac{F_r}{b} = \frac{1}{2} \rho g \left(h_1^2 - h_2^2 \right) + \rho V^2 h_1 - \rho V_2^2 h_2$$

where F_r/b is the reaction force on the gate in the x-direction per unit width of the gate.

To eliminate the velocities from this relation, we use Bernoulli's equation

$$\frac{1}{2} \rho V^2 + p_A + \rho g h_1 = \frac{1}{2} \rho V_2^2 + p_A + \rho g h_2,$$

and using the continuity equation

$$V^2 = \frac{2 g h_2^2}{h_1 + h_2},$$

Substituting into the expression for F_r/b leads to

$$\frac{F_r}{b} = \frac{1}{2} \rho g \frac{(h_1 - h_2)^3}{h_1 + h_2}$$