## Solution to Problem 220D

Applying Bernoulli's equation between a point upstream (point 0) and a point on the surface of the cavity at its maximum radius of $R = R_c$ (point 1) (neglecting gravity)

$$\frac{1}{2}\rho U^2 + p_o = \frac{1}{2}\rho u_1^2 + p_1$$

where $p_1 = p_v$ in order to produce a long, cylindrical cavity. Rearranging

$$p_o - p_v = \frac{1}{2}\rho U^2 \left[ \left(\frac{u_1}{U}\right)^2 - 1 \right]$$

Substituting into the definition of the cavitation number, $\sigma$:

$$\frac{p_o - p_v}{\frac{1}{2}\rho U^2} = \sigma = \left(\frac{u_1}{U}\right)^2 - 1$$

and therefore

$$u_1 = U\sqrt{1+\sigma}$$

Conservation of mass requires that

$$UA_0 = u_1 A_1 \quad \text{or} \quad U\pi R^2 = u_1 \pi \left(R^2 - R_c^2\right)$$

where it is assumed that a negligible amount of mass has been vaporized. Substituting this expression for $u_1$ leads to

$$U\pi R^2 = \left(U\sqrt{1+\sigma}\right)\pi\left(R^2 - R_c^2\right)$$

or

$$\sqrt{1+\sigma} = \frac{R^2}{R^2 - R_c^2}$$

and solving for $\sigma$ yields

$$\sigma = \left(\frac{R^2}{R^2 - R_c^2}\right)^2 - 1 = \left[\frac{1}{1 - \left(\frac{R_c}{R}\right)^2}\right]^2 - 1$$

Using a control volume surrounding all the liquid, the body and the cavity, the momentum theorem in the x-direction yields

$$F_x = \rho u_1 A_1 (u_1) + \rho U A_o(-U) = \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

where $F_x$ is the total force on the control volume in the $x$-direction. Two forces contribute: (1) the net pressure force, $F_p$, on the control volume and (2) the external force, $-F_d$, applied to the body to keep it stationary within the control volume (imposed through an imaginary strut not shown in the sketch). This second contribution is the force imposed on the fluid by the body, which is equal and opposite to the drag, $F_d$. Thus

$$F_x = F_p - F_d$$

Now the pressure force, $F_p$, is given by

$$F_p = \underbrace{p_o \pi R^2}_{\substack{\text{pressure force acting}\\ \text{on left side of CV}}} - \underbrace{p_v \pi R^2}_{\substack{\text{pressure force acting}\\ \text{on right side of CV}}} = (p_o - p_v)\,\pi R^2$$

Using this it follows that

$$F_d = (p_o - p_v)\,\pi R^2 - \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

Substituting for $p_o - p_v$ yields

$$F_d = \frac{1}{2}\rho U^2 \sigma \pi R^2 - \rho\pi R^2 U^2 \left(\sqrt{\sigma+1} - 1\right)$$

which upon simplification yields

$$F_d = \pi\rho U^2 R^2 \left(\frac{\sigma}{2} - \sqrt{\sigma+1} + 1\right)$$