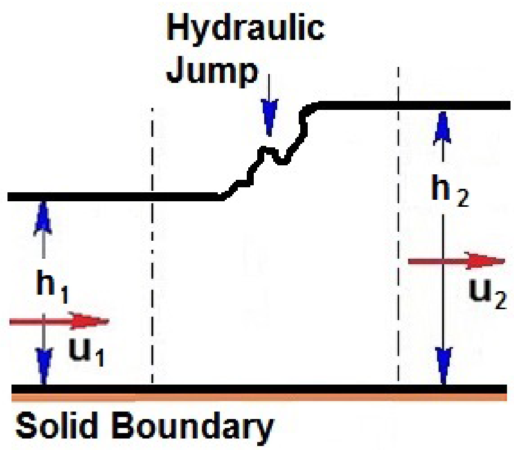
Solution to Problem 220F:



To analyze the details of this phenomena it is most useful to do so in a frame of reference fixed in the jump as shown in the above figure. In this frame of reference, the depth-averaged velocities upstream and downstream of the jump are denoted by u_1 and u_2 respectively and the upstream and downstream depths are denoted by h_1 and h_2. Then we apply the equations of conservation of mass and the linear momentum theorem to a control volume consisting of the upstream and downstream boundaries shown in the figure that extend far above the liquid and the solid boundary at the base of the flow.

Then, assuming for simplicity that the breadth of the flow normal to the sketch is unity, conservation of mass for this flow which is steady in the frame of reference chosen requires that

$$u_1 h_1 \;=\; u_2 h_2 \;=\; Q \tag{1}$$

where Q is used to denote the volume flow rate (per unit breadth). To apply the linear momentum theorem in the direction of flow we note that the hydrostatic force on the upstream boundary is $\rho g h_1^2/2$ (where g is the acceleration due to gravity and ρ is the liquid density assumed constant) while that on the left side is $\rho g h_2^2/2$ so that the net hydrostatic force is $\rho g(h_1^2 - h_2^2)/2$ in the direction of flow. We neglect any viscous forces that may act at the solid boundary. The momentum flux in through the left hand boundary is $\rho u_1^2 h_1$ and out through the right-hand boundary is $\rho u_2^2 h_2$. Consequently the linear momentum theorem yields

$$\rho u_2^2 h_2 - \rho u_1^2 h_1 \;=\; \frac{1}{2}\rho g(h_1^2 - h_2^2) \tag{2}$$

which can be written as

$$\frac{Q^2}{h_1 h_2}(h_1 - h_2) \;=\; \frac{g}{2}(h_1 - h_2)(h_1 + h_2) \tag{3}$$

Therefore *either* $h_1 = h_2$ and there is no jump at all *or*

$$\frac{2Q^2}{g h_1 h_2} \;=\; (h_1 + h_2) \tag{4}$$

which is a quadratic in h_2 and whose solution is

$$h_2 \;=\; -\frac{h_1}{2} \pm \left[\frac{h_1^2}{4} + \frac{8Q^2}{g h_1}\right]^{\frac{1}{2}} \tag{5}$$

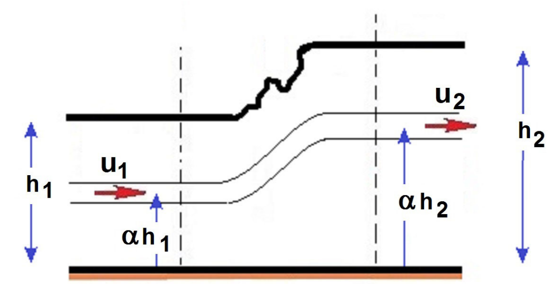
Clearly the only realistic choice is the positive sign and so the downstream depth must be given by

$$h_2 = -\frac{h_1}{2}\left[-1+\left\{1+\frac{8Q^2}{gh_1^3}\right\}^{\frac{1}{2}}\right] \tag{6}$$

This relation between the upstream and downstream depths (or between the upstream and downstream velocities since $h_2/h_1 = u_1/u_2$) can be written in non-dimensional terms using $Fr_1 = u_1/(gh_1)^{1/2}$ and $Fr_2 = u_2/(gh_2)^{1/2}$:

$$Fr_2 = Fr_1\left[\frac{-1+(1+8Fr_1^2)^{\frac{1}{2}}}{2}\right]^{-\frac{3}{2}} \tag{7}$$

To evaluate the energy losses in the flow through a hydraulic jump consider a streamtube through the jump as exemplified by that shown in the following figure: If we denote the atmospheric pressure above



the free surface by p_A, then the static pressure, p_1, in the tube that enters the jump at an elevation of αh_1 $(0 < \alpha < 1)$ will be $p_A + (1-\alpha)\rho g h_1$ and therefore the total pressure, p_1^T, will be

$$p_1^T = p_A + (1-\alpha)\rho g h_1 + \alpha\rho g h_1 + \frac{1}{2}\rho u_1^2 = p_A + \rho g h_1 + \frac{1}{2}\rho u_1^2 \tag{8}$$

using the solid boundary as the elevation reference level. Similarly the total pressure in the streamtube downstream of the jump will be

$$p_2^T = p_A + \rho g h_2 + \frac{1}{2}\rho u_2^2 \tag{9}$$

so the total pressure drop in the stream tube as it passes through the hydraulic jump will be

$$p_1^T - p_2^T = \rho g(h_1 - h_2) + \frac{1}{2}\rho(u_1^2 - u_2^2) \tag{10}$$

and is the same for any streamtube since it is independent of the chosen location, α. Using $Q = u_1h_1 = u_2h_2$ and the above expression for Q this may be written as

$$p_1^T - p_2^T = \frac{\rho g(h_2-h_1)^3}{4h_2h_1} \tag{11}$$

and therefore the energy loss in the hydraulic jump per unit breadth, $\dot{W} = (p_1^T - p_2^T)Q$, is given by

$$\frac{\dot{W}}{\rho g h_1 Q} = \frac{(h_2-h_1)^3}{4h_2h_1^2} \tag{12}$$

This dimensionless energy loss demonstrates the dramatic rise in the energy loss as the magnitude of the jump increases. In most practical circumstances this energy loss manifests itself as massive turbulence and mixing within the jump though very small scale jumps can be quite smooth as the energy is dissipated by viscous effects within the jump.

It is evident from the above relations

- that h_2 must always be greater than h_1.
- The upstream Froude number is always greater than unity.
- The downstream Froude number is always less than one, so that just like a shock wave in a compressible fluid in which the upstream flow relative to the shock is always supersonic and the downstream flow relative to the shock is always subsonic, in a hydraulic jump the upstream flow relative to the jump is always supercritical whereas the downstream flow relative to the jump is always subcritical.
- There is always a dissipation of kinetic energy within the hydraulic jump just as there is a dissipation of kinetic energy in a compressible fluid shock wave. This dissipation of energy occurs through the viscous dissipation and turbulence that always takes place within a hydraulic jump.