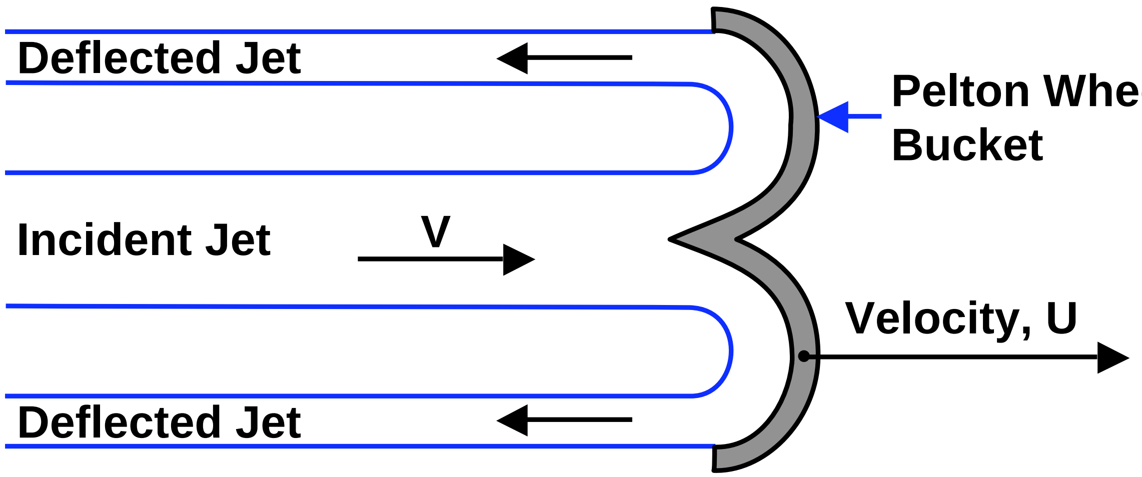
Solution to Problem 220I

In a frame of reference in which the bucket is at rest, the incident jet has a steady velocity $(V - U)$. Neglecting gravity and



viscous effects, Bernoulli's equation along the streamline connecting the incident jet to one of the deflected jets yields:

$$p_1 + \frac{1}{2}\rho(V-U)^2 = p_2 + \frac{1}{2}\rho v_2^2 = p_3 + \frac{1}{2}\rho v_3^2$$

and, since $p_1 = p_2 = p_3 = p_{atm}$, it follows that

$$v_2^2 = v_3^2 = (V-U)^2$$
$$v_2 = v_3 = V - U$$

Additionally, by continuity and symmetry $A_2 = A_3 = A/2$.

From the momentum theorem in the U direction, the force on the bucket, F, in the direction U is given by

$$-F \quad = \quad (V-U)\rho[-(V-U)]A + (-v_2)\rho v_2 A_2 + (-v_3)\rho v_3 A_3$$

so that

$$F = 2\rho A(V-U)^2$$

The power, P, transmitted to the bucket and the turbine wheel is then FU where

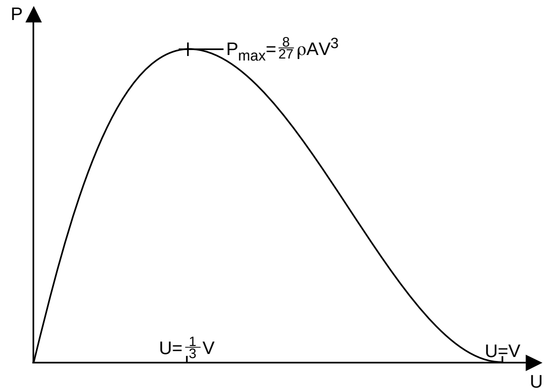
$$P = FU = 2\rho AU(V-U)^2$$

To find the velocity, U_{max}, for which P is a maximum we find that

$$\frac{\partial P}{\partial U} = 2\rho A(V-U)^2 - 4\rho AU(V-U)$$

is zero when $(V-U) - 2U = 0$ and therefore

$$U_{max} = \frac{V}{3}$$



The efficiency, η, as defined is

$$\eta = \frac{P}{\rho AV^3} = \frac{2\rho AU(V-U)^2}{\rho AV^3} = 2\frac{U}{V}\left(1-\frac{U}{V}\right)^2$$

and therefore the efficiency when $U = V/3$, namely the maximum efficiency, is given by

$$\eta|_{U=V/3} = \frac{2}{3}\left(\frac{2}{3}\right)^2 = \frac{8}{27} \approx .296$$