Solution to Problem 222B

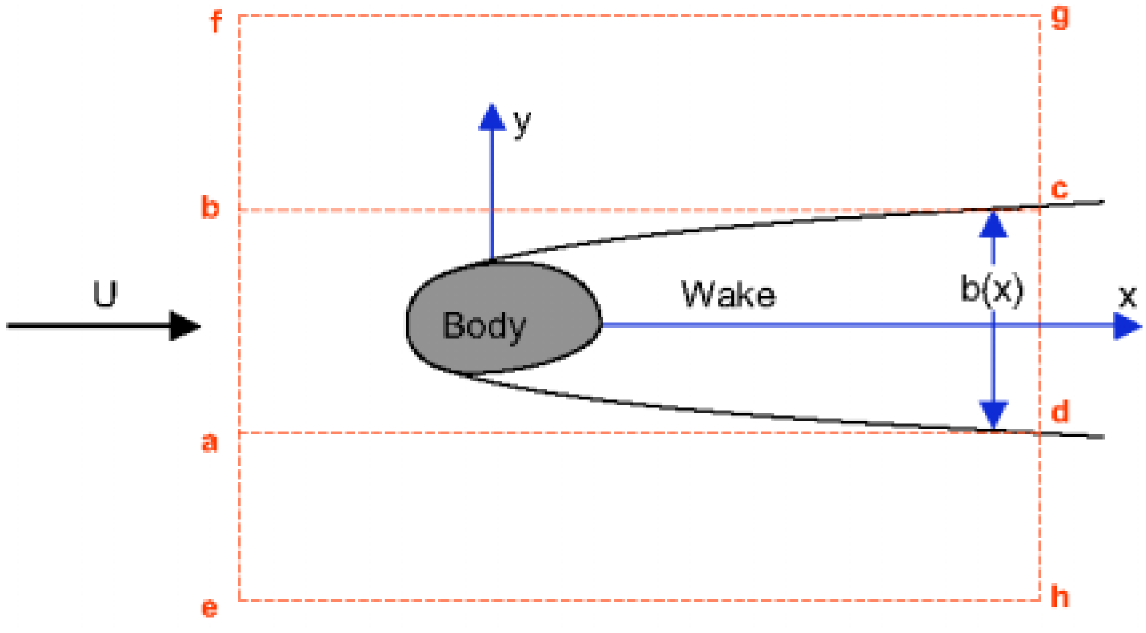

Figure 1: Control volume *abcd* shown by red dashed line.

We choose to utilize the control volume $efgh$ shown above and to evaluate fluxes per unit depth normal to the sketch. Then evaluating the mass flow rates through each of the boundaries of this control volume (CV), the equation of continuity requires that

$$\int_{\mathrm{e}}^{\mathrm{f}} \rho U dy = \int_{\mathrm{g}}^{\mathrm{h}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx$$

where ρ is the fluid density, u and v are the velocity components in the x and y directions. The boundaries ef, fg and he are assumed to be very far away and the boundary gh sufficently far away so that the pressure on all these boundaries is atmospheric pressure and the u velocity on ef, fg and he is U, the free stream velocity far upstream. It is also assumed that the u velocity on the boundary gh *outside of the wake* in regions gc and dh is equal to U. It follows that

$$\int_{\mathrm{e}}^{\mathrm{a}} \rho u dy = \int_{\mathrm{e}}^{\mathrm{a}} \rho U dy = \int_{\mathrm{h}}^{\mathrm{d}} \rho u dy$$

and

$$\int_{\mathrm{b}}^{\mathrm{f}} \rho u dy = \int_{\mathrm{b}}^{\mathrm{f}} \rho U dy = \int_{\mathrm{c}}^{\mathrm{g}} \rho u dy$$

and consequently these parts of the integrals in the first equation cancel so that the first equation becomes

$$\int_{\mathrm{a}}^{\mathrm{b}} \rho U dy = \int_{\mathrm{d}}^{\mathrm{c}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx,$$

So that the first equation can be written as

$$\int_{\mathrm{a}}^{\mathrm{b}} \rho U dy = \int_{\mathrm{d}}^{\mathrm{c}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx,$$

Then, assuming the flow is symmetric about the x axis:

$$\int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} \rho U dy = \int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} \rho u dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v dx - \int_{\mathrm{e}}^{\mathrm{h}} \rho v dx.$$

Rearranging

$$\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx = \int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} (U-u) dy = \int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} A(x) \cos\left[\frac{\pi y}{b(x)}\right] dy$$

$$= A(x)\frac{b(x)}{\pi}\left(\sin\left[\frac{\pi y}{b(x)}\right]\right)_{y=-\frac{b(x)}{2}}^{y=\frac{b(x)}{2}}$$

$$= A(x)\frac{b(x)}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]$$

and therefore

$$\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx = 2A(x)\frac{b(x)}{\pi}$$

We now apply the momentum thereom in the x direction to obtain the force F_x on the control volume $efgh$ per unit depth normal to the sketch as:

$$F_x = \int_{\mathrm{e}}^{\mathrm{f}} \rho(-U)(U) dy + \int_{\mathrm{h}}^{\mathrm{g}} \rho u(u) dy + \int_{\mathrm{f}}^{\mathrm{g}} \rho v(U) dx + \int_{\mathrm{a}}^{\mathrm{d}} \rho(-v)(U) dx$$

As with continuity equation the integrals over bf and cg cancel as do the integrals over cg and hd and this canellation leads to

$$F_x = -\rho U^2 b(x) + \rho \int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} \left\{U - A(x)\cos\left[\frac{\pi u}{b(x)}\right]\right\}^2 dy + \rho U\left[\int_{\mathrm{f}}^{\mathrm{g}} v dx - \int_{\mathrm{e}}^{\mathrm{h}} v dx\right]$$

and substituting for the terms in the square brackets from the reult obtained from the continuity equation we obtain:

$$F_x = -\rho U^2 b(x) + \rho \int_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} \left\{U^2 - 2UA(x)\cos\left[\frac{\pi y}{b(x)}\right] + A(x)^2\cos^2\left[\frac{\pi y}{b(x)}\right]\right\} dy + \rho U\left[2A(x)\frac{b(x)}{\pi}\right]$$

$$= -\rho U^2 b(x) + \rho U^2 b(x) - 2\rho U A(x)\frac{b(x)}{\pi}\sin\left[\frac{\pi y}{b(x)}\right]_{-\frac{b(x)}{2}}^{\frac{b(x)}{2}} + \rho A(x)^2\left\{\frac{y}{2} + \frac{\sin\left[\frac{2\pi y}{b(x)}\right]}{4\frac{\pi}{b(x)}}\right\}_{y=-\frac{b(x)}{2}}^{y=\frac{b(x)}{2}} + 2\rho U A(x)\frac{b(x)}{\pi}$$

$$= -2\rho U A(x)\frac{b(x)}{\pi} + \frac{1}{2}\rho A(x)^2 b(x) = \rho A(x) b(x)\left[\frac{A(x)}{2} - 2\frac{U}{\pi}\right]$$

Because the pressure on all sides of the CV $efgh$ is atmospheric pressure it follows that the contribution of the pressures on the external boundaries to F_x is zero. As a result, the only force on the control volume is the force imposed to keep the body in place a strut not shown in the sketch. This force is equal and opposite to the drag force on the body, D, or $D = -F_x$ and therefore

$$b(x) = \frac{2\pi D}{\rho A(x)\left[4U - \pi A(x)\right]}$$