## Solution to Problem 222C

Though you were not asked to do this, we show first that $U^*$ is indeed the average velocity, $\bar{u}$, of the emerging jet:

$$\bar{u} = \frac{1}{\pi r_0^2}\int_0^{r_0} u(r) 2\pi r dr = \frac{4U^*}{r_0^2}\left[\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right]_0^{r_0} = U^*$$

This problem should be addressed using modifications to the jet engine anlysis for the case of a jet with uniform velocity which is described in the text. For convenience we reproduce the entire, modified analysis here.

Consider the sketch of the cross-section of a jet engine as shown in the figure below. The outline of the throughflow is shown by the dashed blue lines. Far upstream the cross-sectional area of this streamtube is denoted by $A_i$ and the velocity within it is the same as the rest of the upstream flow, namely $U$. Far downstream the cross-section of the jet emerging from the engine is denoted by $A_j$. It is assumed that this jet is axisymmetric and contains the velocity distribution given in the statement of the problem. It is also assumed that any mixing between the jet and the surrounding fluid can be neglected. Viscous effects in the exterior flow are also neglected so that, by Bernoulli's law, the velocity of the fluid exterior to the jet must be $U$.

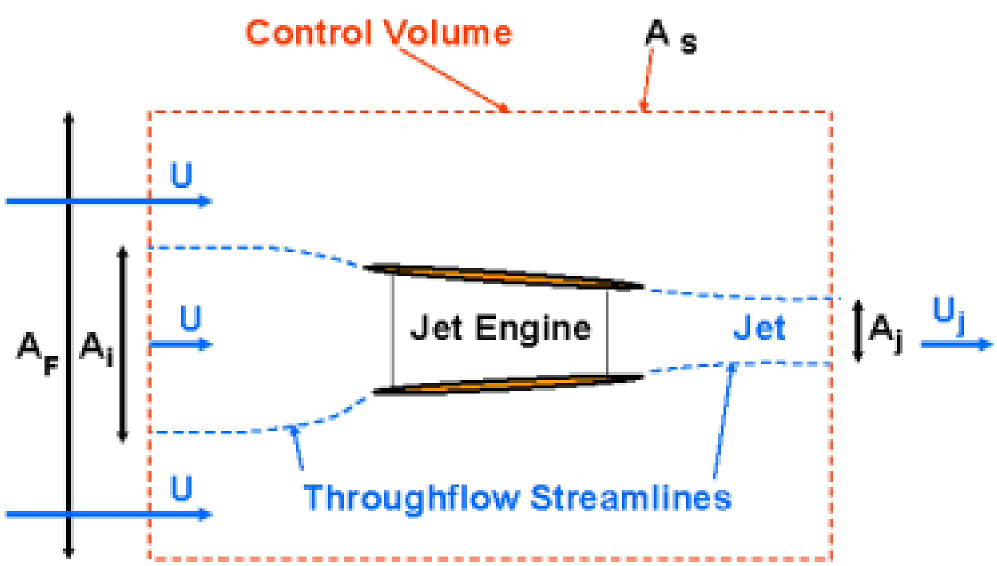


Figure 1: A cross-section through a jet engine showing the outline of the throughflow (dashed blue lines) and a cylindrical control volume (dashed red lines).

We define a large cylindrical control volume as shown by the dashed red lines. The various components of the surface of this control volume are:

- A large upstream area, $A_F$, normal to the oncoming stream which is sufficiently far from the engine so that the pressure on that surface is essentially the atmospheric pressure far upstream.

- Within $A_F$, the intersection of the throughflow streamtube with that area is denoted by $A_i$.
- A large cylindrical surface, $A_S$, which is everywhere parallel with $U$ that represents the outer boundary of the control volume.
- A downstream surface normal to the oncoming stream which is the other end of the cylidrical control volume and therefore also has an area $A_F$.
- Within this downstream area is the circular intersection of the jet which has an area $A_j = \pi r_0^2$.

Therefore, except for the jet area, all the flows on the boundaries of this control volume have a velocity in the $U$ direction equal to $U$; in contrast, the exiting jet has the given velocity distribution. Since the pressures on all the boundaries of the control volume are assumed to be equal to the upstream atmospheric pressure it follows that the densities of the entering and exiting flows are as follows. Except for the exiting jet the other entering and exiting flows have the same density as the upstream flow which will be denoted by $\rho$. In contrast since the temperature of the exiting jet may be much hotter, its density will be denoted by $\rho_j$ and this will be assumed to be uniform across the jet.

With these definitions we can now apply conservation of mass and then the momentum theorem in the $U$ direction. The mass rate at which fuel is added to the flow inside the engine is usually very small compared with the throughflow mass flow rate of the air and so we neglect the added fuel mass (the primary effect of the fuel is to add heat by its combustion). Then, assuming that the flow is steady so that the mass of fluid inside the control volume is unchanging then conservation of mass requires that

$$\rho U A = \rho U(A - A_j) + \rho_j U^* A_j + \dot{M}$$

where $\dot{M}$ denotes the mass flow out through the sides of the control volume, namely the area $A_S$. It follows that

$$\dot{M} = A_j(\rho U - \rho_j U^*)$$

Now we apply the momentum thereom in the $U$ direction to obtain the total force $F$ acting on the contents of the control volume (which includes the jet engine and the jet) in the $U$ direction:

$$F = -\rho U^2 A + \rho U^2(A - A_j) + U\dot{M} + \int_0^{r_0} \rho_j (u(r))^2 2\pi r dr$$

With cancellations and substitution for $\dot{M}$ from the expression derived from continuity, this becomes

$$F = -\rho_j U U^* A_j + \int_0^{r_0} \rho_j (u(r))^2 2\pi r dr$$

$$F = -\rho_j U U^* A_j + \int_0^{r_0} \rho_e 2U[1 - r^2/r_0^2]^2 2\pi r dr$$

$$F = \frac{4\pi}{3}\rho_j r_0^2 U^2 - \rho_j U U^* A_j = \frac{4}{3}\rho_j A_j U^{*2} - \rho_j U U^* A_j = \dot{M}_j \left[\frac{4}{3}U^* - U\right]$$

where $\dot{M}_j = \rho_j U^* A_j$ is the mass flow rate of the flow through the engine.

Finally we must consider the various possible contributions to the total force, $F$, acting on the control volume and its contents in the $U$ direction. It is assumed that the flow has a sufficiently high Reynolds number so that the shear stresses acting on $A_0$ are negigible and so that there are no significant viscous contributions to the normal stresses on the surfaces normal to $U$. Thus the only pertinent forces acting on the external surface of the control volume are those due to the pressure. Moreover it is assumed that these surfaces are sufficiently far from the body that the pressures on all surfaces are equal to the pressure in the uniform stream. It follows that there is no contribution of the pressures to $F$. Consequently if we neglect contributions from body forces such as gravity (or assume $U$ is horizontal), the only contribution to $F$ is the force that must be applied to the body to hold it in place. That force will be the thrust produced by the engine, $T$, defined as the force imposed by the engine on the rest of the airplane (or supporting structure) and acting in the *negative* $U$direction. It follows that the force, $F$, on the engine and therefore on the control volume is equal to $T$ in the *positive* $U$ direction. Therefore $F = T$ and

$$T = \dot{M}_j \left[ \frac{4}{3} U^* - U \right]$$

Note how the thrust is greater than that produced with a uniform jet velocity, $U^*$, because of the factor of $4/3$ in the above equation. With a uniform jet velocity that factor would be unity.

Using the data given the thrust $T$ for the case of the parabolic velocity distribution is calculated to be $7000N$. In contrast the thrust with a uniform jet velocity is $5000N$.