## Solution to Problem 230A

**Continuity:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Since the flow is planar and incompressible this simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since the velocity, $v$, normal to the plate is zero everywhere in the flow it follows from continuity that

$$\frac{\partial u}{\partial x} = 0$$

so $u$ is only a function of $y$, $u = u(y)$.

**Navier-Stokes:**
**x-direction:**

$$\rho \left[ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Since the flow is planar, since $v = 0$ and $\frac{\partial u}{\partial x} = 0$, and since the pressure is constant, this becomes:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{d^2 u}{dy^2}$$

We use separation of variables to solve this partial differential equation. Assume

$$u(y,t) = Y(y)T(t)$$

Substituting this into the partial differential equation and rearranging, the result can be written as a term which is a function only of $y$ equal to a term which is a function only of $t$. It follows that both must be equal to a simple constant, $\lambda$:

$$\frac{1}{T}\frac{dT}{dt} = \frac{\mu}{\rho}\frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda$$

The equation for $t$ is then:

$$\frac{dT}{dt} = \lambda T$$

and the solution to this is:

$$T(t) = c_1 e^{\lambda t}$$

The equation for $y$ is:

$$\frac{d^2 Y}{dy^2} - \frac{\rho}{\mu} Y = 0$$

and the solution to this is:

$$Y(y) = c_2 e^{\sqrt{\rho\lambda/\mu}\ y} + c_3 e^{-\sqrt{\rho\lambda/\mu}\ y}$$

The boundary conditions at the plate and as $y \rightarrow \infty$ are respectively

$$u(0,t) = U(t) = Ue^{kt}$$

and

$$u(y \rightarrow \infty, t) = 0$$

The second condition yields $c_2 = 0$. It follows that the solution for $u(y,t)$ is:

$$u(y,t) = c_4 e^{-\sqrt{\rho\lambda/\mu}\ y}\ e^{\lambda t}$$

where $c_4 = c_1 c_3$. Applying the no-slip boundary condition at the surface of the plate:

$$u(0,t) = c_4 e^{\lambda t} = U e^{kt}$$

so the values of the unknown constant $c_4 = U$ and $\lambda = k$ are now determined. This yields a velocity profile:

$$u(y,t) = U e^{kt} e^{-\sqrt{k/\nu}\ y}$$

where $\nu$ is the kinematic viscosity $\nu = \mu/\rho$. The vorticity, $\omega(y,t)$, is given by

$$\omega(\mathbf{y},\mathbf{t}) = \nabla \times \mathbf{u} = -\frac{\partial u}{\partial y}$$

$$\omega = U\sqrt{\frac{k}{\nu}}\ e^{kt} e^{-\sqrt{\rho k/\mu}\ y}$$

The boundary layer thickness, $\delta$, is defined as that distance from the plate where the velocity is 10% of the plate velocity:

$$\begin{aligned} 0.1\ U e^{kt} &= U e^{kt} e^{-\sqrt{k/\nu}\ \delta} \\ 0.1 &= e^{-\sqrt{k/\nu}\ \delta} \\ \delta &= \ln(10)\sqrt{\frac{\nu}{k}} \end{aligned}$$