Solution to Problem 267B

Note that this is a continuation of Problem 267A. We denote the non-dimensional parameters and variables by an overbar so that

$$\tilde{\omega} = \omega\delta/U \quad , \quad \tilde{k} = k\delta \quad , \quad \tilde{y} = y\delta \quad , \quad \tilde{f} = f/\delta U$$

so that the Rayleigh equation can be written as

$$(\tilde{\omega} - \tilde{k}\tilde{u}) \left[\frac{d^2\tilde{f}}{d\tilde{y}^2} - \tilde{k}^2\tilde{f} \right] + \tilde{k}\frac{d^2\tilde{u}}{d\tilde{y}^2}\tilde{f} = 0$$

In this specific problem the velocity profile $\tilde{u}$ is given by

$$\begin{aligned} \tilde{u} &= \tilde{y} + \tilde{y}^2 - \tilde{y}^3 \quad \text{for } 0 < \tilde{y} < 1 \\ &= 1 \quad \text{for } \tilde{y} > 1 \end{aligned}$$

so that

$$\begin{aligned} \frac{d^2\tilde{u}}{d\tilde{y}^2} &= 2 - 6\tilde{y} \quad \text{for } 0 < \tilde{y} < 1 \\ &= 0 \quad \text{for } \tilde{y} > 1 \end{aligned}$$

Appropriate boundary conditions are that $\tilde{f}(0 = \tilde{f}(\infty) = 0$ so that the perturbation velocity in the y-direction, $\partial\psi/\partial y$, is zero both on the solid surface, $y = 0$ and at $y = \infty$. We also need to impose continuity and smoothness at the edge of the boundary layer, so that $\tilde{f}(1-) = \tilde{f}(1+)$ and $\partial\tilde{f}/\partial\tilde{y}(1-) = \partial\tilde{f}/\partial\tilde{y}(1+)$.

Note that the specified velocity profile has a point of inflexion at $\tilde{y} = 1/3$ which, according to Rayleigh and Tollmein, is a necessary and sufficient condition for temporal instability (though perhaps not spatial instability).

To solve the above problem numerically, we first note that for $\tilde{y} > 1$ the stability equation reduces to

$$\frac{d^2\tilde{f}}{d\tilde{y}^2} - \tilde{k}^2\tilde{f} = 0$$

whose general solution has the form

$$\tilde{f}(\tilde{y}) = A \exp(\tilde{k}\tilde{y}) + B \exp(-\tilde{k}\tilde{y})$$

and since the disturbance is required to die out as $\tilde{y} \to \infty$ we set $A = 0$. Moreover since the eigenfunctions (the solution) will contain an arbitrary multiplicative constant we are free to set $B = 1$. Therefore $\tilde{f}(1+)$ and $\partial\tilde{f}/\partial\tilde{y}(1+)$ are known and therefore $\tilde{f}(1-)$ and $\partial\tilde{f}/\partial\tilde{y}(1-)$ are also known. Consequently we can begin the integration of the Rayleigh equation at $\tilde{y} = 1$ using these values of $\tilde{f}(1)$ and $\partial\tilde{f}/\partial\tilde{y}(1)$ and initial or adjusted values of $\tilde{k}_R$ and $\tilde{k}_I$ (for a given $\tilde{\omega}$) and proceed to integrate down to $\tilde{y} = 0$. We then test whether the boundary condition $\tilde{y}(0) = 0$ is satisfied (real and imaginery parts both must be zero). If not we readjust $\tilde{k}_R$ and $\tilde{k}_I$ and in this way iterate toward the solution.