Solution to Problem 268A

Consider the momentum flux caused by fluctuations only.

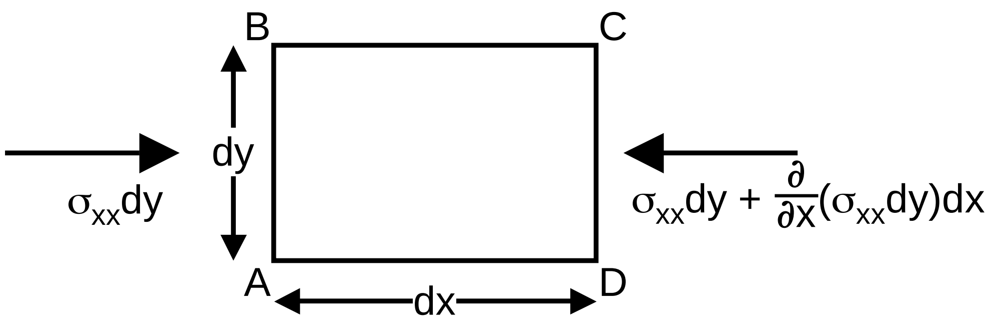

Figure 1: control volume

The fluctuating momentum flux is given by the total mass flux ($\dot{m} = \rho(\bar{u}+u')dy$) multiplied by the fluctuating velocity. Flux of x-momentum in through AB is:

$$\dot{m}u' = \rho u' dy(\overline{u} + u')$$

Flux of x-momentum out through CD is:

$$\overline{m}u' + \frac{\partial}{\partial x}\left[\overline{m}u'\right] dx = \rho u' dy(\overline{u} + u') + \frac{\partial}{\partial x}\left[\rho u' dy\left(\overline{u} + u'\right)\right] dx$$

So, the net flux of x-momentum out through AB and CD is:

$$\rho dx dy \frac{\partial}{\partial x}\left[u'\left(\overline{u} + u'\right)\right]$$

To get the time-average momentum flux caused by this fluctuating momentum flux, the previous equation will be averaged keeping in mind

$$\overline{u'} = 0, \quad \overline{\overline{u}u'} = 0, \quad \overline{u'u'} \neq 0$$

yielding for the average momentum flux caused by the fluctuating velocities

$$dx dy \frac{\partial}{\partial x}(\rho\overline{u'^2})$$

The conceptual forces diagram due to the momentum flux of fluctuations is shown in figure 1.
So the net additional normal force is:

$$\begin{aligned} \Sigma F_x &= -\frac{\partial \sigma_{xx}}{\partial x} \\ -dx dy \frac{\partial \sigma_{xx}}{\partial x} &= dx dy \frac{\partial}{\partial x}(\rho\overline{u'^2}) \end{aligned}$$

so the net additional normal stress, the Reynolds stress, is

$$\sigma_{xx} = -\rho\overline{u'^2}$$