Solution to Problem 270B

For a pipe of radius R (and diameter D, so that $D = 2R$), the friction factor, f, is defined as

$$f = \frac{8\tau_w}{\rho V^2}$$

where V is the average velocity through the pipe,

$$V = \frac{Q}{\pi R^2}$$

where

$$\begin{aligned} Q &= 2\pi \int_0^R \bar{u}(r) r dr \\ &= 2\pi \int_0^R \bar{u}(y)\,(R-y)\,dy \end{aligned}$$

is the volume flow rate. Of note is the fact that the velocity profile is given in y, the distance from the outside of the pipe and the radius r is a measure of the distance from the center out. The average velocity is thus

$$\begin{aligned} V &= \frac{2}{R^2}\int_0^R \bar{u}(y)\,(R-y)\,dy \\ &= \frac{2}{R^2}u_\tau \int_0^R u^*(y^*)\,(R-y)\,dy \\ &= \frac{2}{R^2}u_\tau \int_0^R 8.7\left(\frac{u_\tau y}{\nu}\right)^{\frac{1}{7}}(R-y)\,dy \\ &= \frac{(2)(8.7)}{R^2}u_\tau \left[\frac{7}{8}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} R y^{\frac{8}{7}} - \frac{7}{15}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} y^{\frac{15}{7}}\right]_0^R \\ &= \frac{(2)(8.7)}{R^2}u_\tau \left[\frac{49}{120}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} R^{\frac{15}{7}}\right] \\ &= \frac{(2)(8.7)(49)}{120}u_\tau \left(\frac{u_\tau R}{\nu}\right)^{\frac{1}{7}} \\ &= 7.105\left(\frac{R}{\nu}\right)^{\frac{1}{7}} u_\tau^{\frac{8}{7}} \end{aligned}$$

where

$$u^* = \frac{\bar{u}}{u_\tau} = 8.7\,(y^*)^{\frac{1}{7}} = 8.7\left(\frac{u_\tau y}{\nu}\right)^{\frac{1}{7}}$$

Substituting $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ and $f = \frac{8\tau_w}{\rho V^2}$ into the average velocity yields

$$\begin{aligned} V &= 7.105\sqrt{\frac{\tau_w}{\rho}}\left[\left(\sqrt{\frac{\tau_w}{\rho}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \\ &= 7.105\sqrt{\frac{fV^2}{8}}\left[\left(\sqrt{\frac{fV^2}{8}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \\ \frac{1}{7.105} &= \sqrt{\frac{f}{8}}\left[\left(\sqrt{\frac{fV^2}{8}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \end{aligned}$$

$$
\begin{aligned}
\frac{\sqrt{8}}{7.105} &= \sqrt{f}\left[\left(\sqrt{\frac{f}{8}}\right)\frac{VR}{\nu}\right]^{\frac{1}{7}} \\
\frac{\sqrt{8}\,(8)^{\frac{1}{14}}}{7.105} &= \sqrt{f}\left[\left(\sqrt{f}\right)\frac{VD}{2\nu}\right]^{\frac{1}{7}} \\
\frac{\sqrt{8}\,(8)^{\frac{1}{14}}\,(2)^{\frac{1}{7}}}{7.105} &= f^{\frac{8}{14}}\left(\frac{VD}{\nu}\right)^{\frac{1}{7}} \\
f^{\frac{4}{7}} &= \frac{\sqrt{8}\,(8)^{\frac{1}{14}}\,(2)^{\frac{1}{7}}}{7.105}\mathrm{Re}^{-\frac{1}{7}} \\
f &\simeq 0.308\ \mathrm{Re}^{-\frac{1}{4}},
\end{aligned}
$$

where

$$
\mathrm{Re} = \frac{VD}{\nu}
$$

For $\mathrm{Re} = 1 \times 10^6$, this equation yields a friction factor for smooth pipes of $f = .00974$ compared to the value given by the graph of $f = 0.0117$. Thus the equation under-predicts the friction factor by approximately 18%.