Solution to Problem 270E

The shear stress σ_{xy} is composed of contributions from the Reynolds shear stress and the laminar viscous stress so that

$$\sigma_{xy} = \mu \left(\frac{d\overline{u}}{dy} \right) - \rho \overline{u'v'}$$

or, using Prandtl's mixing length model with a universal constant, κ:

$$\sigma_{xy} = \mu \left(\frac{d\overline{u}}{dy} \right) + \rho \kappa^2 y^2 \left(\frac{d\overline{u}}{dy} \right)^2$$

But since the shear stress in a cylindrical pipe must vary linearly with the radius

$$\sigma_{xy} = (\sigma_{xy})_{y=0} \frac{r}{R} = (\sigma_{xy})_{y=0} \frac{R-y}{R} = \frac{\tau_w (R-y)}{R}$$

Therefore the differential equation for $\overline{u}(y)$ becomes

$$\left(\frac{d\overline{u}}{dy} \right)^2 + \frac{\nu}{\kappa^2 y^2} \left(\frac{d\overline{u}}{dy} \right) - \frac{\tau_w (R-y)}{\rho \kappa^2 y^2 R} = 0$$

or, solving the quadratic for $(d\overline{u}/dy)$,

$$\left(\frac{d\overline{u}}{dy} \right) = \frac{1}{2\rho\kappa^2 y^2} \left[\left(\mu^2 + 4\rho\kappa^2 y^2 \tau_w (1 - y/R) \right)^{\frac{1}{2}} - \mu \right]$$

We note parenthetically that unless the viscous stress is included the velocity gradient becomes infinite at the wall. However, we could continue toward a solution for the flow outside the laminar sublayer on the wall by setting $\mu = 0$ to obtain

$$\left(\frac{d\overline{u}}{d(r/R)} \right) = -\frac{1}{\kappa^2} \left(\frac{\tau_w}{\rho} \right)^{\frac{1}{2}} \left(\frac{r}{R} \right)^{\frac{1}{2}} \frac{1}{(1 - r/R)}$$

This can be integrated to find $\overline{u}(r/R)$:

$$\overline{u} \left(\frac{r}{R} \right) = \frac{1}{\kappa^2} \left(\frac{\tau_w}{\rho} \right)^{\frac{1}{2}} \left[2 \left(\frac{r}{R} \right)^{\frac{1}{2}} + ln \left(\frac{\left(\frac{r}{R}\right)^{\frac{1}{2}} - 1}{\left(\frac{r}{R}\right)^{\frac{1}{2}} + 1} \right) \right] + C$$

where C is an integration constant determined by matching this result with the velocity at the edge of the laminar sub-layer.