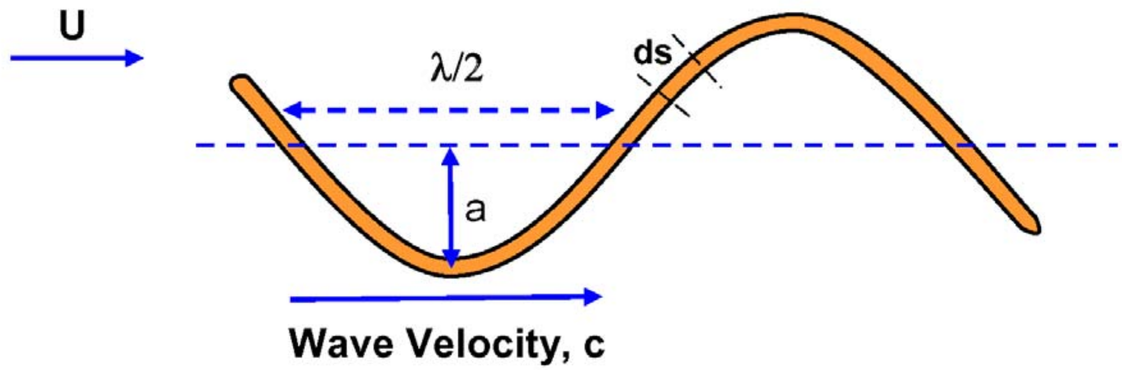
Solution to Problem 295D

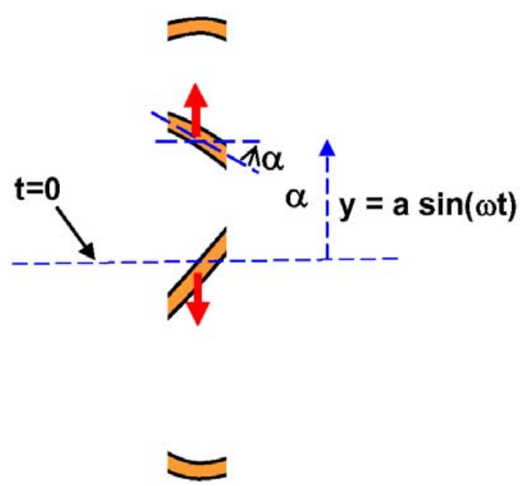
First we make a Galilean coordinate shift so that the flagellum as a whole is not moving but the liquid is approaching with a free stream velocity, U: Then we focus on just one material element of the flagellum and recognize the movement of that



short section, ds, during the transit of the traveling wave deformation. For convenience set the origin of time, $t = 0$ as that moment when the element ds is passing upwards through its middle location (the axis shown by the dashed horizontal line). Since the wave is traveling to the right the inclination of the element at this moment $t = 0$ is from the upper left to the lower right. Then, as the element moves on, we denote its vertical displacement, y, by

$$y = a \sin \omega t$$

where, as stipulated, the amplitude of the traveling wave is a and the frequency, $\omega = 2\pi c/\lambda$ where c is the wave velocity relative to the material of the flagellum and λ is the wavelength. We denote the inclination of the element to the horizontal

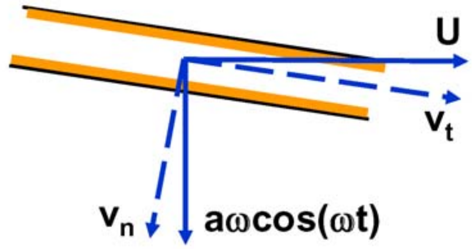


by α (positive clockwise) where

$$\tan \alpha = \frac{\omega a}{c} \cos \omega t$$

Then it follows that the velocities of the liquid relative to the element are given by U in the horizontal direction and $a\omega \cos \omega t$ in the downward vertical direction as shown in the following sketch: To continue we resolve these relative velocities in directions normal and tangential to the inclination of the element as shown above and obtain:

$$v_n = a\omega \cos \omega t \cos \alpha - U \sin \alpha$$



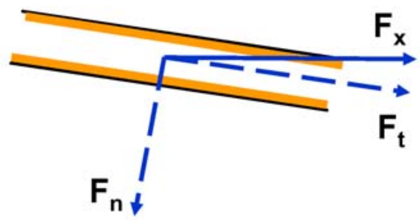
$$v_t = a\omega \cos \omega t \sin \alpha + U \cos \alpha$$

We now invoke the low Reynolds number resistance relations for cylindrical elements namely that the force on the element in the direction normal to the axis of the element, F_n, is equal to the normal coefficient, k_n, times v_n and the force on the element in the direction tangential to the axis of the element, F_t, is equal to the tangential coefficient, k_t, times v_t or

$$F_n = k_n(a\omega \cos \omega t \cos \alpha - U \sin \alpha)$$

$$F_t = k_t(a\omega \cos \omega t \sin \alpha + U \cos \alpha)$$

as shown below. We will also assume that $k_n = 2k_t$.



Since we are concerned with forward propulsion of the microorganism we now evaluate the force on the element in the x or forward direction, F_x:

$$F_x = F_t \cos \alpha - F_n \sin \alpha$$

$$F_x = \frac{k_t \left[U + \left(\frac{2U}{c} - 1\right) \frac{\omega^2 a^2}{c} \cos^2 \omega t\right]}{1 + \frac{\omega^2 a^2}{c^2} \cos^2 \omega t}$$

using preceding relations. Since the organism is self-propelling the time-averaged value of this force must be zero and therefore the relation between the parameters for a self-propelling organism is

$$\left[1 + \frac{\left(1 - \frac{c}{U}\right) \cos^2 \omega t}{\frac{c^2}{\omega^2 a^2} + \cos^2 \omega t}\right]_{\text{time average}} = 0$$

This is the expression that we must solve to find the speed of propulsion, U, of the organism. This can be solved numerically to obtain the ratio, U/c, as a function of the amplitude parameter, $\omega a/c$. An approximate solution using the fact that the mean value of $\cos^2 \omega t$ is $1/2$ is

$$\frac{U}{c} = \frac{1}{2\left(1 + \frac{c^2}{\omega^2 a^2}\right)}$$