Solution to Problem 295E

Find the velocity at which the amoeba moves through the liquid.

Since the amoeba is moving at very low Reynolds number, we can use Stokes' Stream Function.

$$\psi = \sin^2\theta \left[-\frac{Ur^2}{2} + \frac{A}{r} + Br \right]$$

This leads to velocity components in the radial and tangential directions:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} = \cos\theta \left[-U + \frac{2A}{r^3} + \frac{2B}{r} \right]$$

$$u_\theta = -\frac{1}{r\sin\theta} \frac{\partial\psi}{\partial r} = \sin\theta \left[U + \frac{A}{r^3} - \frac{B}{r} \right]$$

The first boundary condition on the flow around the amoeba states that no flow can penetrate the surface of the organism:

$$u_r(r = R) = 0$$

The second boundary condition demands that there be no slip of the flow at the amoeba's surface. Since the amoeba is regenerating its surface to create a surface velocity of $U^* \sin\theta$, we have the following condition.

$$u_\theta(r = R) = U^* \sin\theta$$

This is enough information to solve for the constants A and B, but to relate U and U^* we need another condition on the motion of the amoeba. Since the amoeba is moving at a constant velocity in the horizontal direction, the sum of forces in this direction must be equal to zero. To get the total force, we need to integrate the shear and normal stresses over the surface of the organism.

In axisymmetric, spherical coordinates the shear stress, $\tau_{r\theta}$, is given by:

$$\tau_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

Evaluating the shear stress with the velocity components from the Strokes' Stream Function:

$$\begin{aligned} \tau_{r\theta} &= \mu \sin\theta \left(\frac{U}{r} - 2\frac{A}{r^4} - 2\frac{B}{r^2} - 3\frac{A}{r^4} + \frac{B}{r^2} - \frac{U}{r} - \frac{A}{r^4} + \frac{B}{r^2} \right) \\ &= -6\mu \sin\theta \frac{A}{r^4} \end{aligned}$$

To find the normal stress, we integrate the equations of motion in their simplified form for low Reynolds number:

$$\nabla p = \mu \nabla^2 \vec{u}$$

In the radial direction, this gives us:

$$\frac{\partial p}{\partial r} = \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u_r}{\partial\theta} \right) - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial(u_\theta \sin\theta)}{\partial\theta} \right]$$

Evaluating this equation with the velocity components of the Stokes' Stream Function:

$$\begin{aligned} \frac{\partial p}{\partial r} &= \mu \cos\theta \left(12\frac{A}{r^5} + 2\frac{U}{r^2} - 4\frac{A}{r^5} - 4\frac{B}{r^3} + 2\frac{U}{r^2} - 4\frac{A}{r^5} - 4\frac{B}{r^3} - 4\frac{U}{r^2} - 4\frac{A}{r^5} + 4\frac{B}{r^3} \right) \\ &= -4\mu \cos\theta \frac{B}{r^3} \end{aligned}$$

Integrating:

$$p(r,\theta) = \int \left(-4\mu\cos\theta\frac{B}{r^3}\right)\partial r = 2\frac{\mu B}{r^2}\cos\theta + f(\theta)$$

Repeating this procedure in the tangential direction leads to:

$$p(r,\theta) = \int \left(-2\mu\sin\theta\frac{B}{r^2}\right)\partial\theta = 2\frac{\mu B}{r^2}\cos\theta + g(r)$$

So $f = g =$ constant; Far from the amoeba ($r \to \infty$), the pressure is equal to the ambient pressure, p_∞, so the pressure distribution around the amoeba is given by:

$$p(r,\theta) = p_\infty + 2\frac{\mu B}{r^2}\cos\theta$$

The net force in the horizontal direction is given by:

$$F_x = -\int_0^\pi \tau_{r\theta}|_{r=R}\sin\theta dA - \int_0^\pi p|_{r=R}\cos\theta dA$$

where $dA = 2\pi R^2\sin\theta d\theta$. Evaluating each of the integrals:

$$\begin{aligned}
\int_0^\pi \tau_{r\theta}|_{r=R}\sin\theta dA &= -12\pi R^2\frac{\mu A}{R^4}\int_0^\pi \sin^3\theta d\theta \\
&= -\frac{4}{3}\pi R^2\frac{12\mu A}{R^4} \\
\int_0^\pi p|_{r=R}\cos\theta dA &= 2\pi R^2\int_0^\pi \left(p_\infty\sin\theta\cos\theta + 2\frac{\mu B}{R^2}\sin\theta\cos^2\theta\right)d\theta \\
&= \frac{4}{3}\pi R^2\frac{4\mu B}{R^2}
\end{aligned}$$

$$\Rightarrow F_x = \frac{4}{3}\pi R^2\mu\left(12\frac{A}{R^4} - 4\frac{B}{R^2}\right)$$

Setting this equal to zero since there is no net force on the amoeba, we get a relationship between the constants A and B.

$$A = \frac{1}{3}BR^2$$

The no-slip condition ($u_\theta(r=R) = U^*\sin\theta$) gives:

$$\sin\theta\left(U + \frac{A}{R^3} - \frac{B}{R}\right) = U^*\sin\theta$$

$$\Rightarrow B = \frac{3}{2}R(U - U^*)$$

The impenetrable condition ($u_r(r=R) = 0$) gives:

$$\cos\theta\left(-U + 2\frac{A}{R^3} + 2\frac{B}{R}\right)$$

$$\Rightarrow U = 2\frac{A}{R^3} + 2\frac{B}{R} \tag{1}$$

Combining above equations gives the velocity at which the amoeba moves through the liquid.

$$U = \frac{2}{3}U^*$$