Solution to Problem 312B:

Heat is being added to the steady, frictionless flow of a perfect gas (ratio of specific heats, γ) in a pipe of constant, uniform cross-sectional area. The speed of sound and Mach number of the flow are denoted by c and M respectively and vary with position, x, measured along the pipe. If the rate of heat addition is Q per unit time per unit length of the pipe and the mass flow rate of gas is denoted by m we seek to find an expression for dM/dx in terms of Q, m, γ, c and M.

The heat added to unit mass in a distance, dx, is Qdx/m and this must be equal to the increase in total enthalpy for a unit mass so

$$\frac{Qdx}{m} = c_p\, dT + u\, du \tag{1}$$

and therefore from the energy equation

$$\frac{\gamma \mathcal{R} T}{(\gamma-1)}\frac{dT}{T} + u^2\frac{du}{u} = \frac{Qdx}{m} \tag{2}$$

or

$$\frac{1}{(\gamma-1)}\frac{dT}{T} + M^2\frac{du}{u} = \frac{Qdx}{mc^2} \tag{3}$$

But for frictionless flow in a constant, uniform duct, the continuity equation becomes

$$\frac{du}{u} + \frac{d\rho}{\rho} = 0 \tag{4}$$

and the momentum equation yields:

$$\frac{dp}{\rho} + u\, du = 0 \quad \text{or} \quad \frac{dp}{p} = -\gamma M^2 \frac{du}{u} \tag{5}$$

The perfect gas law yields

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{6}$$

and combining the three above equations yields

$$\frac{dT}{T} = (1-\gamma M^2)\frac{du}{u} \tag{7}$$

and substituting this into the second equation produces

$$\frac{du}{u} = \frac{(\gamma-1)}{(1-M^2)}\frac{Q}{mc^2}dx \tag{8}$$

But also since $M^2 = u^2/\gamma \mathcal{R} T$ it follows that

$$MdM = M^2\frac{du}{u} - M^2\frac{dT}{2T} \tag{9}$$

and using the above relations

$$\frac{dM}{dx} = \frac{M(1+\gamma M^2)(\gamma-1)}{2(1-M^2)}\frac{Q}{mc^2} \tag{10}$$

Note that dM/dx takes a different sign depending upon whether the flow is subsonic or supersonic.