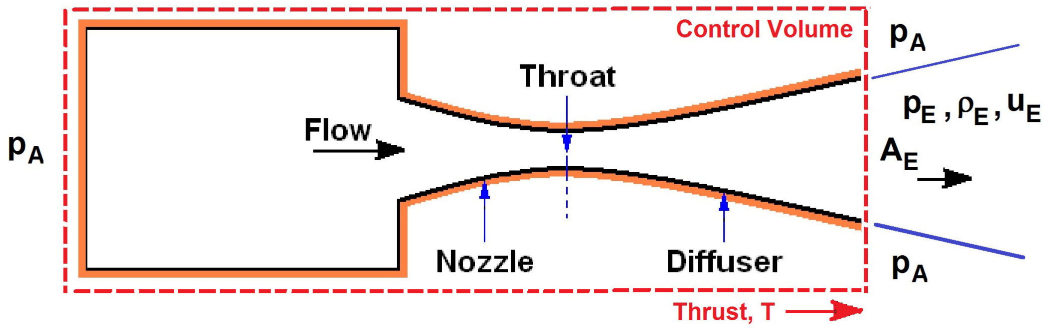
Solution to Problem 324A:

A rocket engine is modeled by a reservoir of gas at high temperature and pressure (p_0) feeding gas to a convergent/divergent nozzle:



The flow reaches critical, choked conditions at the throat (Area A^*), is subsonic upstream of the throat and supersonic in the divergent section between the throat and the exit. The pressure at exit (Area A_E) is p_E and the surrounding atmospheric pressure is p_A We wish to find an expression for the thrust produced by the engine in terms of p_0, p_E, p_A, A^*, A_E and γ (the ratio of specific heats).

By the momentum theorem the net force on the engine in the jet direction is equal to Thrust $-A_E(p_E-p_A)$ where the object is the thrust produced by the engine and this must equal the flux of momentum out of the control volume, $\rho_E A_E u_E^2$, so

$$\text{Thrust} \;=\; \rho_E A_E u_E^2 + A_E(p_E - p_A) \tag{1}$$

It remains to evaluate $\rho_E A_E u_E^2$ in terms of the nozzle flow. Actually you were given *more* information than you need since, given p_0, A_0 and A^* and the presumption of completely isentropic flow, the jet exit pressure, p_E, should follow from these quantities and the answer for the thrust does not need to include p_E. In other words, given the inherent relation for p_E in terms of p_0, A_0 and A^*, the answer to the question can take several alternative forms and it is instructive to explore two of these equivalent forms:

First approach: Continuity requires that $\rho_E A_E u_E = \rho^* A^* u^*$ and with $M^* = 1$ so $u^* = (\gamma \mathcal{R} T^*)^{1/2}$ and $p^* = \gamma \mathcal{R} T^*$ it follows that

$$\rho_E A_E u_E^2 \;=\; \frac{\rho^*}{\rho_E}\frac{A^{*2}}{A_E}\,\gamma p^* \tag{2}$$

But isentropic relations require that $\rho^*/\rho_E = (p^*/p_E)^{1/\gamma}$ and the choked conditions in the throat mean that $p^*/p_0 = (2/(\gamma+1))^{\gamma/(\gamma-1)}$ so that

$$\rho_E A_E u_E^2 \;=\; \frac{\gamma A^{*2}}{A_E}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{p_0}{p_E}\right)^{1/\gamma} p_0 \tag{3}$$

and hence

$$\text{Thrust} \;=\; \frac{\gamma A^{*2}}{A_E}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{p_0}{p_E}\right)^{1/\gamma} p_0 + A_E(p_E - p_0) \tag{4}$$

Alternative derivation: By noting that in any isentropic flow

$$\left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} = 1 + \frac{(\gamma-1)}{2} M^2 \tag{5}$$

and that $M^2 = u^2/\gamma\mathcal{R}T = \rho u^2/\gamma p$ it follows that

$$\rho u^2 = \frac{2\gamma p}{(\gamma-1)} \left[\left(\frac{p_0}{p_E}\right)^{(\gamma-1)/\gamma} - 1\right] \tag{6}$$

and therefore

$$\rho_E A_E u_E^2 = \frac{2\gamma p_E A_E}{(\gamma-1)} \left[\left(\frac{p_0}{p_E}\right)^{(\gamma-1)/\gamma} - 1\right] \tag{7}$$

and therefore

$$\text{Thrust} = \frac{2\gamma p_E A_E}{(\gamma-1)} \left[\left(\frac{p_0}{p_E}\right)^{(\gamma-1)/\gamma} - 1\right] + A_E(p_E - p_0) \tag{8}$$

The two answers are equivalent since

$$\left[\frac{\gamma+1}{2}\right]^{(\gamma+1)/(\gamma-1)} \left(\frac{p_E}{p_0}\right)^{(\gamma+1)/\gamma} \left[\left(\frac{p_0}{p_E}\right)^{(\gamma-1)/\gamma} - 1\right] = \frac{(\gamma-1)}{2} \frac{A^{*2}}{A_E^2} \tag{9}$$

a relation which can be deduced from continuity, the isentropic relations and the choked nozzle condition.