Solution to Problem 330A

For convencience, we shift the frame of reference from a shock moving at speed, u_s, to a stationary shock with oncoming velocity, $u_1 = u_s$. We then apply conservation of mass, momemtum, and energy across the shock and the equation of state to get the desired relation for the shock speed.

State:

$$p = \rho RT$$

Mass:

$$\rho_1 u_1 = \rho_2 u_2$$

Momentum:

$$p_1 + \rho u_1^2 = p_2 + \rho_2 u_2^2$$

Using mass to replace $\rho_2 u_2$ with $\rho_1 u_1$:

$$\Rightarrow \frac{p_1 - p_2}{\rho_1 u_1} = u_2 - u_1$$

Energy:

$$h_1 + \frac{1}{2}U_1^2 = h_2 + \frac{1}{2}U_2^2$$

Assuming a calorically perfect gas, we take $h = c_p T$.

$$c_p T_1 + \frac{1}{2}u_1^2 = c_p T_2 + \frac{1}{2}u_2^2$$

Given the relationship between c_p, γ, and R: $c_p = \frac{\gamma R}{\gamma - 1}$ we have:

$$\frac{\gamma R}{\gamma - 1}T_1 + \frac{1}{2}u_1^2 = \frac{\gamma R}{\gamma - 1}T_2 + \frac{1}{2}u_2^2$$

Using the equation of state to replace temerature in favor of density and pressure:

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

Using the combination of momentum and mass from above, the previous equation can be rewritten as:

$$\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} = \frac{\gamma - 1}{2\gamma}\left(2u_1 + \frac{p_1 - p_2}{\rho_1 u_1}\right)\left(\frac{p_1 - p_2}{\rho_1 u_1}\right)$$

Using mass:

$$p_1 - \frac{u_2}{u_1}p_2 = \frac{\gamma - 1}{2\gamma}\left(2 + \frac{p_1 - p_2}{\rho_1 u_1^2}\right)(p_1 - p_2)$$

Using the momemtum-mass combination to eliminate u_2:

$$p_1 - \left(1 + \frac{p_1 - p_2}{\rho_1 u_1^2}\right)p_2 = \frac{\gamma - 1}{2\gamma}\left(2 + \frac{p_1 - p_2}{\rho_1 u_1^2}\right)(p_1 - p_2)$$

Now introducing the definition of the speed of sound: $c^2 = \frac{\gamma p}{\rho}$:

$$p_1 - \left(1 + \frac{p_1 - p_2}{\frac{\gamma p_1}{c_1^2}u_1^2}\right)p_2 = \frac{\gamma - 1}{2\gamma}\left(2 + \frac{p_1 - p_2}{\frac{\gamma p_1}{c_1^2}u_1^2}\right)(p_1 - p_2)$$

Forming the Mach number, $M = \frac{u}{c}$ and rewriting:

$$(p_1 - p_2)\left(1 - \frac{p_2}{\gamma p_1 M_1^2}\right) = \frac{\gamma - 1}{2\gamma}\left(2 + \frac{1 - \frac{p_2}{p_1}}{\gamma M_1^2}\right)(p_1 - p_2)$$

$$\Rightarrow M_1^2 = \frac{p_2}{p_1} - \frac{\gamma - 1}{2\gamma}\frac{p_2}{p_1} + \frac{\gamma - 1}{2\gamma}$$

Simplifying:

$$u_s^2 = c_1^2\left[\frac{\gamma - 1}{2\gamma} + \frac{\gamma + 1}{2\gamma}\frac{p_2}{p_1}\right]$$

$$\Rightarrow u_s = c_1\sqrt{\frac{\gamma - 1}{2\gamma} + \frac{\gamma + 1}{2\gamma}\frac{p_2}{p_1}}$$

Calculate the speed if the temperature of the ambient air is 30°C and the pressure ratio, p_2/p_1, is 3.0.

$$c_1 = \sqrt{\gamma R T_1}$$

$$\Rightarrow u_s = \sqrt{(1.4)(280)(303)}\sqrt{\frac{1.4 - 1}{2(1.4)} + \frac{1 + 1.4}{2(1.4)}(3)} = 567.8\frac{m}{s}$$