Introduction to Free Streamline Flows

In these sections we briefly survey the extensive literature on fully developed cavity flows and the methods used for their solution. The terms "free streamline flow" or "free surface flow" are used for those situations that involve a "free" surface whose location is initially unknown and must be found as a part of the solution. In the context of some of the multiphase flow literature, they would be referred to as *separated flows*. In the introduction to multiphase flows we described the two asymptotic states of a multiphase flow, *homogeneous* and *separated* flow. Sections (Nl) and (Nm) describe some of the homogeneous flow methods and their application to cavitating flows; this chapter presents the other approach. However, we shall not use the term *separated flow* in this context because of the obvious confusion with the accepted, fluid mechanical use of the term.

Fully developed cavity flows constitute one subset of free surface flows, and this survey is intended to provide information on some of the basic properties of these flows as well as the methods that have been used to generate analytical solutions of them. A number of excellent reviews of free streamline methods can be found in the literature, including those of Birkkoff and Zarantonello (1957), Parkin (1959), Gilbarg (1960), Woods (1961), Gurevich (1961), Sedov (1966), and Wu (1969, 1972). Here we shall follow the simple and elegant treatment of Wu (1969, 1972).

The subject of free streamline methods has an interesting history, for one can trace its origins to the work of Kirchhoff (1869), who first proposed the idea of a "wake" bounded by free streamlines as a model for the flow behind a finite, bluff body. He used the mathematical methods of Helmholtz (1868) to find the irrotational solution for a flat plate set normal to an oncoming stream. The pressure in the wake was assumed to be constant and equal to the upstream pressure. Under these conditions (the zero cavitation number solution described below) the wake extends infinitely far downstream of the body. The drag on the body is nonzero, and Kirchhoff proposed this as the solution to D'Alembert's paradox (see section (Neb)), thus generating much interest in these free streamline methods, which Levi-Civita (1907) later extended to bodies with curved surfaces. It is interesting to note that Kirchhoff's work appeared many years before Prandtl discovered boundary layers and the reason for the wake structure behind a body. However, Kirchhoff made no mention of the possible application of his methods to cavity flows; indeed, the existence of these flows does not seem to have been recognized until many years later.

In these sections (Nu) we focus on the application of free streamline methods to fully developed cavity flows; for a modern view of their application to wake flows the reader is referred to Wu (1969, 1972). It is important to take note of the fact that, because of its low density relative to that of the liquid, the nature of the vapor or gas in the fully developed cavity usually has little effect on the liquid flow. Thus the pressure gradients due to motion of the vapor/gas are normally negligible relative to the pressure gradients in the liquid, and consequently it is usually accurate to assume that the pressure, p_c, acting on the free surface is constant. Similarly, the shear stress that the vapor/gas imposes on the free surface is usually negligible. Moreover, other than the effect on p_c, it is of little consequence whether the cavity contains vapor or noncondensable gas, and the effect of p_c is readily accommodated in the context of free streamline flows by defining the cavitation number, σ, as

$$\sigma = \frac{p_\infty - p_c}{\frac{1}{2}\rho_L U_\infty^2} \qquad \text{(Nua1)}$$

where p_c has replaced the p_V of the previous definition and we may consider p_c to be due to any combination of vapor and gas. It follows that the same free streamline analysis is applicable whether the cavity is a true vapor cavity or whether the wake has been filled with noncondensable gas externally introduced into the "cavity." The formation of such gas-filled wakes is known as "ventilation." Ventilated cavities can occur either because of deliberate air injection into a wake or cavity, or they may occur in the ocean due to naturally occurring communication between, say, a propeller blade wake and the atmosphere above the ocean surface. For a survey of ventilation phenomena the reader is referred to Acosta (1973).

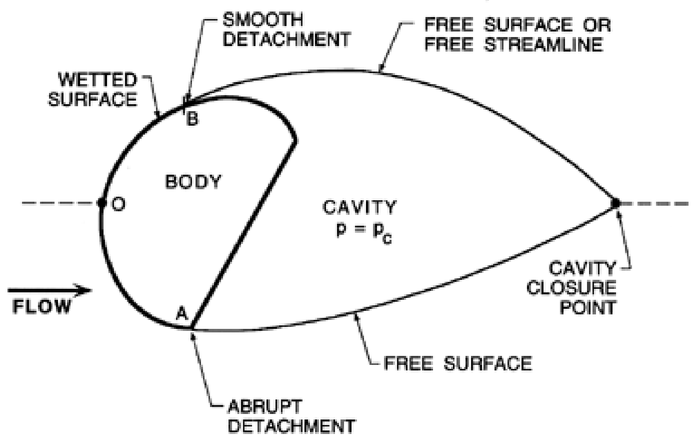

Figure 1: Schematic showing the terminology used in the free streamline analysis.

Most of the available free streamline methods assume inviscid, irrotational and incompressible flow, and comparisons with experimental data suggest, as we shall see, that these are reasonable approximations. Viscous effects in fully developed cavity flows are usually negligible so long as the free streamline detachment locations (see Figure 1) are fixed by the geometry of the body. The most significant discrepancies occur when detachment is not fixed but is located at some initially unknown point on a smooth surface (see section (Nuc)). Then differences between the calculated and observed detachment locations can cause substantial discrepancies in the results.

Assuming incompressible and irrotational flow, the problems require solution of Laplace's equation for the velocity potential, $\phi(x_i, t)$,

$$\nabla^2 \phi = 0 \tag{Nua2}$$

subject to the following boundary conditions:

1. On a solid surface, $\mathcal{S}_W(x_i, t)$, the kinematic condition of no flow through that surface requires that

$$\frac{d\mathcal{S}_W}{dt} = \frac{\partial \mathcal{S}_W}{\partial t} + (\nabla \phi) \cdot \nabla \mathcal{S}_W = 0 \tag{Nua3}$$

2. On a free surface, $\mathcal{S}_F(x_i, t)$, a similar kinematic condition that neglects the liquid evaporation rate yields

$$\frac{d\mathcal{S}_F}{dt} = \frac{\partial \mathcal{S}_F}{\partial t} + (\nabla \phi) \cdot \nabla \mathcal{S}_F = 0 \tag{Nua4}$$

3. Assuming that the pressure in the cavity, p_c, is uniform and constant, leads to an additional dynamic boundary condition on $\mathcal{S}_F$. Clearly, the dimensionless equivalent of p_c, namely σ, is a basic parameter in this class of problem and must be specified *a priori*. In steady flow, neglecting surface tension and gravitational effects, the magnitude of the velocity on the free surface, q_c, should be uniform and equal to $U_\infty(1 + \sigma)^{\frac{1}{2}}$.

The two conditions on the free surface create serious modeling problems both at the detachment points and in the cavity closure region (Figure 1). These issues will the addressed in the two sections that follow.

In planar, two-dimensional flows the powerful methods of complex variables and the properties of analytic functions (see, for example, Churchill 1948) can be used with great effect to obtain solutions to these irrotational flows (see the review articles and books mentioned above). Indeed, the vast majority of the published literature is devoted to such methods and, in particular, to steady, incompressible, planar potential flows. Under those circumstances the complex velocity potential, f, and the complex conjugate velocity, w, defined by

$$f = \phi + i\psi \quad ; \quad w = \frac{df}{dz} = u - iv \tag{Nua5}$$

are both analytic functions of the position vector $z = x + iy$ in the physical, (x, y) plane of the flow. In this context it is conventional to use i rather than j to denote $(-1)^{\frac{1}{2}}$ and we adopt this notation. It follows that the solution to a particular flow problem consists of determining the form of the function, $f(z)$ or $w(z)$. Often this takes a parametric form in which $f(\zeta)$ (or $w(\zeta)$) and $z(\zeta)$ are found as functions of some parametric variable, $\zeta = \xi + i\eta$. Another very useful device is the logarithmic hodograph variable, ϖ, defined by

$$\varpi = \log \frac{q_c}{w} = \chi + i\theta \quad ; \quad \chi = \ln \frac{q_c}{|w|} \quad ; \quad \theta = \tan^{-1} \frac{v}{u} \tag{Nua6}$$

The value of this variable lies in the fact that its real part is known on a free surface, whereas its imaginary part is known on a solid surface.