Particle Equation of Motion

In a multiphase flow with a very dilute discrete phase the fluid forces discussed in sections (Nea) to (Neh) will determine the motion of the particles that constitute that discrete phase. In this section we discuss the implications of some of the fluid force terms. The equation that determines the particle velocity, V_i, is generated by equating the total force, F_i^T, on the particle to $m_p dV_i/dt^*$. Consider the motion of a spherical particle (or bubble) of mass m_p and volume v (radius R) in a *uniformly* accelerating fluid. The simplest example of this is the vertical motion of a particle under gravity, g, in a pool of otherwise quiescent fluid. Thus the results will be written in terms of the buoyancy force. However, the same results apply to motion generated by any uniform acceleration of the fluid, and hence g can be interpreted as a general uniform fluid acceleration (dU/dt). This will also allow some tentative conclusions to be drawn concerning the relative motion of a particle in the nonuniformly accelerating fluid situations that can occur in general multiphase flow. For the motion of a sphere at small relative Reynolds number, $Re \ll 1$ (where $Re = 2WR/\nu_C$ and W is the typical magnitude of the relative velocity), only the forces due to buoyancy and the weight of the particle need be added to F_i as given by equations (Neh17) or (Neh21) in order to obtain F_i^T. This addition is simply given by $(\rho_C v - m_p)g_i$ where g is a vector in the vertically upward direction with magnitude equal to the acceleration due to gravity. On the other hand, at high relative Reynolds numbers, $Re \gg 1$, one must resort to a more heuristic approach in which the fluid forces given by equation (Neg18) are supplemented by drag (and lift) forces given by $\frac{1}{2}\rho_C A C_{ij}|W_j|W_j$ as in equation (Nee3). In either case it is useful to nondimensionalize the resulting equation of motion so that the pertinent nondimensional parameters can be identified.

Examine first the case in which the relative velocity, W (defined as positive in the direction of the acceleration, g, and therefore positive in the vertically upward direction of the rising bubble or sedimenting particle), is sufficiently small so that the relative Reynolds number is much less than unity. Then, using the Stokes boundary conditions, the equation governing W may be obtained from equation (Neh16) as

$$w + \frac{dw}{dt_*} + \left\{ \frac{9}{\pi(1 + 2m_p/\rho_C v)} \right\}^{\frac{1}{2}} \int_0^{t_*} \frac{dw}{d\tilde{t}} \frac{d\tilde{t}}{(t_* - \tilde{t})^{\frac{1}{2}}} = 1 \qquad \text{(Nei1)}$$

where the dimensionless time, $t_* = t/t_u$ and the relaxation time, t_u, is given by

$$t_u = R^2(1 + 2m_p/\rho_C v)/9\nu_C \qquad \text{(Nei2)}$$

and $w = W/W_\infty$ where W_∞ is the steady terminal velocity given by

$$W_\infty = 2R^2 g(1 - m_p/\rho_C v)/9\nu_C \qquad \text{(Nei3)}$$

In the absence of the Basset term the solution of equation (Nei1) is simply

$$w = 1 - e^{-t/t_u} \qquad \text{(Nei4)}$$

and therefore the typical response time is given by the relaxation time, t_u (see, for example, Rudinger 1969 and section (Nbg)). In the general case that includes the Basset term the dimensionless solution, $w(t_*)$, of equation (Nei1) depends only on the parameter $m_p/\rho_C v$ (particle mass/displaced fluid mass) appearing in the Basset term. Indeed, the dimensionless equation (Nei1) clearly illustrates the fact that the Basset term is much less important for solid particles in a gas where $m_p/\rho_C v \gg 1$ than it is for bubbles in a liquid

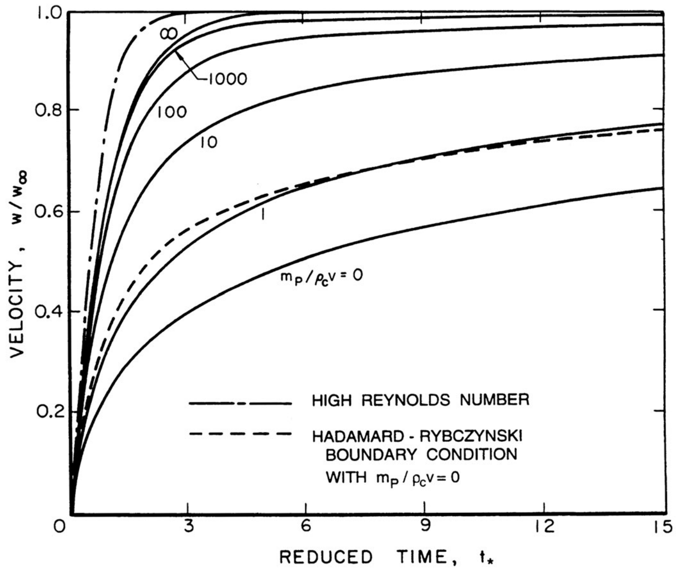

Figure 1: The velocity, W, of a particle released from rest at $t_* = 0$ in a quiescent fluid and its approach to terminal velocity, W_∞. Horizontal axis is a dimensionless time defined in text. Solid lines represent the low Reynolds number solutions for various particle mass/displaced mass ratios, $m_p/\rho_C v$, and the Stokes boundary condition. The dashed line is for the Hadamard-Rybczynski boundary condition and $m_p/\rho_C v = 0$. The dash-dot line is the high Reynolds number result; note that t_* is nondimensionalized differently in that case.

where $m_p/\rho_C v \ll 1$. Note also that for initial conditions of zero relative velocity ($w(0) = 0$) the small-time solution of equation (Nei1) takes the form

$$w = t_* - \frac{2}{\pi^{\frac{1}{2}} \left\{1 + 2m_p/\rho_C v\right\}^{\frac{1}{2}}} t_*^{\frac{3}{2}} + \ldots \qquad \text{(Nei5)}$$

Hence the initial acceleration at $t = 0$ is given dimensionally by

$$2g(1 - m_p/\rho_C v)/(1 + 2m_p/\rho_C v)$$

or $2g$ in the case of a massless bubble and $-g$ in the case of a heavy solid particle in a gas where $m_p \gg \rho_C v$. Note also that the effect of the Basset term is to *reduce* the acceleration of the relative motion, thus increasing the time required to achieve terminal velocity.

Numerical solutions of the form of $w(t_*)$ for various $m_p/\rho_C v$ are shown in figure 1 where the delay caused by the Basset term can be clearly seen. In fact in the later stages of approach to the terminal velocity the Basset term dominates over the added mass term, (dw/dt_*). The integral in the Basset term becomes approximately $2t_*^{\frac{1}{2}} dw/dt_*$ so that the final approach to $w = 1$ can be approximated by

$$w = 1 - C \exp\left\{ -t_*^{\frac{1}{2}} \Big/ \left(\frac{9}{\pi\{1 + 2m_p/\rho_C v\}} \right)^{\frac{1}{2}} \right\} \qquad \text{(Nei6)}$$

where C is a constant. As can be seen in figure 1, the result is a much slower approach to W_∞ for small $m_p/\rho_C v$ than for larger values of this quantity.

The case of a bubble with Hadamard-Rybczynski boundary conditions is very similar except that

$$W_\infty = R^2 g(1 - m_p/\rho_C v)/3\nu_C \qquad \text{(Nei7)}$$

and the equation for $w(t_*)$ is

$$w + \frac{3}{2}\frac{dw}{dt_*} + 2\int_0^{t_*} \frac{dw}{d\tilde{t}}\Gamma(t_* - \tilde{t})d\tilde{t} = 1 \qquad \text{(Nei8)}$$

where the function, $\Gamma(\xi)$, is given by

$$\Gamma(\xi) = \exp\left\{(1 + \frac{2m_p}{\rho_C v})\xi\right\} \operatorname{erfc}\left\{\left((1 + \frac{2m_p}{\rho_C v})\xi\right)^{\frac{1}{2}}\right\} \qquad \text{(Nei9)}$$

For the purposes of comparison the form of $w(t_*)$ for the Hadamard-Rybczynski boundary condition with $m_p/\rho_C v = 0$ is also shown in figure 1. Though the altered Basset term leads to a more rapid approach to terminal velocity than occurs for the Stokes boundary condition, the difference is not qualitatively significant.

If the terminal Reynolds number is much greater than unity then, in the absence of particle growth, equation (Neg18) heuristically supplemented with a drag force of the form of equation (Neg20) leads to the following equation of motion for unidirectional motion:

$$w^2 + \frac{dw}{dt_*} = 1 \qquad \text{(Nei10)}$$

where $w = W/W_\infty, t_* = t/t_u$, and the relaxation time, t_u, is now given by

$$t_u = (1 + 2m_p/\rho_C v)(2R/3C_D g(1 - m_p/v\rho_C))^{\frac{1}{2}} \qquad \text{(Nei11)}$$

and

$$W_\infty = \{8Rg(1 - m_p/\rho_C v)/3C_D\}^{\frac{1}{2}} \qquad \text{(Nei12)}$$

The solution to equation (Nei10) for $w(0) = 0$,

$$w = \tanh t_* \qquad \text{(Nei13)}$$

is also shown in figure 1 though, of course, t_* has a different definition in this case.

The relaxation times given by the expressions (Nei2) and (Nei11) are particularly valuable in assessing relative motion in disperse multiphase flows. When this time is short compared with the typical time associated with the fluid motion, the particle will essentially follow the fluid motion and the techniques of homogeneous flow (see section (Nl)) are applicable. Otherwise the flow is more complex and special effort is needed to evaluate the relative motion and its consequences.

For the purposes of reference in section (Nfa) note that, if we define a Reynolds number, Re, and a Froude number, Fr, by

$$Re = \frac{2W_\infty R}{\nu_C} \quad ; \quad Fr = \frac{W_\infty}{\{2Rg(1 - m_p/\rho_C v)\}^{\frac{1}{2}}} \qquad \text{(Nei14)}$$

then the expressions for the terminal velocities, W_∞, given by equations (Nei3), (Nei7) and (Nei12) can be written as

$$Fr = (Re/18)^{\frac{1}{2}} \ , \ Fr = (Re/12)^{\frac{1}{2}} \ , \quad \text{and} \quad Fr = (4/3C_D)^{\frac{1}{2}} \qquad \text{(Nei15)}$$

respectively. Indeed, dimensional analysis of the governing Navier-Stokes equations requires that the general expression for the terminal velocity can be written as

$$F(Re, Fr) = 0 \qquad \text{(Nei16)}$$

or, alternatively, if C_D is defined as $4/3Fr^2$, then it could be written as

$$F^*(Re, C_D) = 0 \qquad \text{(Nei17)}$$