Unsteady Potential Flow

In general, a particle moving in any flow other than a steady uniform stream will experience fluid accelerations, and it is therefore necessary to consider the structure of the equation governing the particle motion under these circumstances. Of course, this will include the special case of acceleration of a particle in a fluid at rest (or with a steady streaming motion). As in the earlier sections we shall confine the detailed solutions to those for a spherical particle or bubble. Furthermore, we consider only those circumstances in which both the particle and fluid acceleration are in one direction, chosen for convenience to be the x_1 direction. The effect of an external force field such as gravity will be omitted; it can readily be inserted into any of the solutions that follow by the addition of the conventional buoyancy force.

All the solutions discussed are obtained in an accelerating frame of reference *fixed* in the center of the fluid particle. Therefore, if the velocity of the particle in some original, noninertial coordinate system, x_i^*, was $V(t)$ in the x_1^* direction, the Navier-Stokes equations in the new frame, x_i, fixed in the particle center are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_C}\frac{\partial P}{\partial x_i} + \nu_C \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad \text{(Neg1)}$$

where the pseudo-pressure, P, is related to the actual pressure, p, by

$$P = p + \rho_C x_1 \frac{dV}{dt} \quad \text{(Neg2)}$$

Here the conventional time derivative of $V(t)$ is denoted by d/dt, but it should be noted that in the original x_i^* frame it implies a Lagrangian derivative following the particle. As before, the fluid is assumed incompressible (so that continuity requires $\partial u_i/\partial x_i = 0$) and Newtonian. The velocity that the fluid would have at the x_i origin in the absence of the particle is then $W(t)$ in the x_1 direction. It is also convenient to define the quantities r, θ, u_r, u_θ as shown in the figure in section (Neb) and the Stokes streamfunction as in equations (Neb4). In some cases we shall also be able to consider the unsteady effects due to growth of the bubble so the radius is denoted by $R(t)$.

First consider inviscid potential flow for which equations (Neg1) may be integrated to obtain the Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho_C} + \frac{1}{2}(u_\theta^2 + u_r^2) = constant \quad \text{(Neg3)}$$

where ϕ is a velocity potential ($u_i = \partial\phi/\partial x_i$) and ψ must satisfy the equation

$$L\psi = 0 \quad \text{where} \quad L \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \quad \text{(Neg4)}$$

This is of course the same equation as in steady flow and has harmonic solutions, only five of which are necessary for present purposes:

$$\psi = \sin^2\theta\left\{-\frac{Wr^2}{2} + \frac{D}{r}\right\} + \cos\theta\sin^2\theta\left\{\frac{2Ar^3}{3} - \frac{B}{r^2}\right\} + E\cos\theta \quad \text{(Neg5)}$$

$$\phi = \cos\theta\left\{-Wr + \frac{D}{r^2}\right\} + (\cos^2\theta - \frac{1}{3})\left\{Ar^2 + \frac{B}{r^3}\right\} + \frac{E}{r} \quad \text{(Neg6)}$$

$$u_r = \cos\theta \left\{ -W - \frac{2D}{r^3} \right\} + (\cos^2\theta - \frac{1}{3}) \left\{ 2Ar - \frac{3B}{r^4} \right\} - \frac{E}{r^2} \quad \text{(Neg7)}$$

$$u_\theta = -\sin\theta \left\{ -W + \frac{D}{r^3} \right\} - 2\cos\theta\sin\theta \left\{ Ar + \frac{B}{r^4} \right\} \quad \text{(Neg8)}$$

The first part, which involves W and D, is identical to that for steady translation. The second, involving A and B, will provide the fluid velocity gradient in the x_1 direction, and the third, involving E, permits a time-dependent particle (bubble) radius. The W and A terms represent the fluid flow in the absence of the particle, and the D, B, and E terms allow the boundary condition

$$(u_r)_{r=R} = \frac{dR}{dt} \quad \text{(Neg9)}$$

to be satisfied provided

$$D = -\frac{WR^3}{2} \quad , \quad B = \frac{2AR^5}{3} \quad , \quad E = -R^2\frac{dR}{dt} \quad \text{(Neg10)}$$

In the absence of the particle the velocity of the fluid at the origin, $r = 0$, is simply $-W$ in the x_1 direction and the gradient of the velocity $\partial u_1/\partial x_1 = 4A/3$. Hence A is determined from the fluid velocity gradient in the original frame as

$$A = \frac{3}{4}\frac{\partial U}{\partial x_1^*} \quad \text{(Neg11)}$$

Now the force, F_1, on the bubble in the x_1 direction is given by

$$F_1 = -2\pi R^2 \int_0^\pi p \sin\theta\cos\theta d\theta \quad \text{(Neg12)}$$

which upon using equations (Neg2), (Neg3) and (Neg6) to (Neg8) can be integrated to yield

$$\frac{F_1}{2\pi R^2 \rho_C} = -\frac{D}{Dt}(WR) - \frac{4}{3}RWA + \frac{2}{3}R\frac{dV}{dt} \quad \text{(Neg13)}$$

Reverting to the original coordinate system and using v as the sphere volume for convenience ($v = 4\pi R^3/3$), one obtains

$$F_1 = -\frac{1}{2}\rho_C v\frac{dV}{dt^*} + \frac{3}{2}\rho_C v\frac{DU}{Dt^*} + \frac{1}{2}\rho_C(U-V)\frac{dv}{dt^*} \quad \text{(Neg14)}$$

where the two Lagrangian time derivatives are defined by

$$\frac{D}{Dt^*} \equiv \frac{\partial}{\partial t^*} + U\frac{\partial}{\partial x_1^*} \quad \text{(Neg15)}$$

$$\frac{d}{dt^*} \equiv \frac{\partial}{\partial t^*} + V\frac{\partial}{\partial x_1^*} \quad \text{(Neg16)}$$

Equation (Neg14) is an important result, and care must be taken not to confuse the different time derivatives contained in it. Note that in the absence of bubble growth, of viscous drag, and of body forces, the equation of motion that results from setting $F_1 = m_p dV/dt^*$ is

$$\left(1 + \frac{2m_p}{\rho_C v}\right)\frac{dV}{dt^*} = 3\frac{DU}{Dt^*} \quad \text{(Neg17)}$$

where m_p is the mass of the *particle.* Thus for a massless bubble the acceleration of the bubble is three times the fluid acceleration.

In a more comprehensive study of unsteady potential flows Symington (1978) has shown that the result for more general (i.e., noncolinear) accelerations of the fluid and particle is merely the vector equivalent of equation (Neg14):

$$F_i = -\frac{1}{2}\rho_C v \frac{dV_i}{dt^*} + \frac{3}{2}\rho_C v \frac{DU_i}{Dt^*} + \frac{1}{2}\rho_C (U_i - V_i)\frac{dv}{dt^*} \tag{Neg18}$$

where

$$\frac{d}{dt^*} = \frac{\partial}{\partial t^*} + V_j \frac{\partial}{\partial x_j^*} \quad ; \quad \frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + U_j \frac{\partial}{\partial x_j^*} \tag{Neg19}$$

The first term in equation (Neg18) represents the conventional added mass effect due to the particle acceleration. The factor 3/2 in the second term due to the fluid acceleration may initially seem surprising. However, it is made up of two components:

1. $\frac{1}{2}\rho_C dV_i/dt^*$, which is the added mass effect of the fluid acceleration
2. $\rho_C v DU_i/Dt^*$, which is a *buoyancy*-like force due to the pressure gradient associated with the fluid acceleration.

The last term in equation (Neg18) is caused by particle (bubble) volumetric growth, dv/dt^*, and is similar in form to the force on a source in a uniform stream.

Now it is necessary to ask how this force given by equation (Neg18) should be used in the practical construction of an equation of motion for a particle. Frequently, a viscous drag force F_i^D, is quite arbitrarily added to F_i to obtain some total *effective* force on the particle. Drag forces, F_i^D, with the conventional forms

$$F_i^D = \frac{C_D}{2}\rho_C |U_i - V_i|(U_i - V_i)\pi R^2 \qquad (Re \gg 1) \tag{Neg20}$$

$$F_i^D = 6\pi\mu_C (U_i - V_i) R \qquad (Re \ll 1) \tag{Neg21}$$

have both been employed in the literature. It is, however, important to recognize that there is no fundamental analytical justification for such superposition of these forces. At high Reynolds numbers, we noted in the last section that experimentally observed added masses are indeed quite close to those predicted by potential flow within certain parametric regimes, and hence the superposition has some experimental justification. At low Reynolds numbers, it is improper to use the results of the potential flow analysis. The appropriate analysis under these circumstances is examined in the next section.