Dynamic Transfer Functions

As in the quasistatic analyses described at the beginning of this chapter, the first step in the frequency domain approach is to identify all the flow variables that are needed to completely define the state of the flow at each of the nodes of the system. Typical flow variables are the pressure, p, (or total pressure, p^T) the velocities of the phases or components, the volume fractions, and so on. To simplify matters we count only those variables that are not related by simple algebraic expressions. Thus we do not count both the pressure and the density of a phase that behaves barotropically, nor do we count the mixture density, ρ, and the void fraction, α, in a mixture of two incompressible fluids. The minimum number of variables needed to completely define the flow at all of the nodes is called *the order of the system* and will be denoted by N. Then the state of the flow at any node, i, is denoted by the vector of state variables, $\{q_i^n\}, n = 1, 2 \rightarrow N$. For example, in a homogeneous flow we could choose $q_i^1 = p$, $q_i^2 = u$, $q_i^3 = \alpha$, to be the pressure, velocity and void fraction at the node i.

The next step in a frequency domain analysis is to express all the flow variables, $\{q_i^n\}, n = 1, 2 \rightarrow N$, as the sum of a mean component (denoted by an overbar) and a fluctuating component (denoted by a tilde) at a frequency, ω. The complex fluctuating component incorporates both the amplitude and phase of the fluctuation:

$$\{q^n(s,t)\} = \{\bar{q}^n(s)\} + Re\left\{\{\tilde{q}^n(s,\omega)\}e^{i\omega t}\right\} \tag{Nrl1}$$

for $n = 1 \rightarrow N$ where i is $(-1)^{\frac{1}{2}}$ and Re denotes the real part. For example

$$p(s,t) = \bar{p}(s) + Re\left\{\tilde{p}(s,\omega)e^{i\omega t}\right\} \tag{Nrl2}$$

$$\dot{m}(s,t) = \bar{\dot{m}}(s) + Re\left\{\tilde{\dot{m}}(s,\omega)e^{i\omega t}\right\} \tag{Nrl3}$$

$$\alpha(s,t) = \bar{\alpha}(s) + Re\left\{\tilde{\alpha}(s,\omega)e^{i\omega t}\right\} \tag{Nrl4}$$

Since the perturbations are assumed linear ($|\tilde{u}| \ll \bar{u}$, $|\tilde{\dot{m}}| \ll \bar{\dot{m}}$, $|\tilde{q}^n| \ll \bar{q}^n$) they can be readily superimposed, so a summation over many frequencies is implied in the above expressions. In general, the perturbation quantities, $\{\tilde{q}^n\}$, will be functions of the mean flow characteristics as well as position, s, and frequency, ω.

The utilization of transfer functions in the context of fluid systems owes much to the pioneering work of Pipes (1940). The concept is the following. If the quantities at inlet and discharge are denoted by subscripts $m = 1$ and $m = 2$, respectively, then the transfer matrix, $[T]$, is defined as

$$\{\tilde{q}_2^n\} = [T]\{\tilde{q}_1^n\} \tag{Nrl5}$$

It is a square matrix of order N. For example, for an order $N = 2$ system in which the independent fluctuating variables are chosen to be the total pressure, $\tilde{p}^T$, and the mass flow rate, $\tilde{\dot{m}}$, then a convenient transfer matrix is

$$\begin{Bmatrix} \tilde{p}_2^T \\ \tilde{\dot{m}}_2 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \tilde{p}_1^T \\ \tilde{\dot{m}}_1 \end{Bmatrix} \tag{Nrl6}$$

In general, the transfer matrix will be a function of the frequency, ω, of the perturbations and the mean flow conditions in the device. Given the transfer functions for each component one can then synthesize transfer functions for the entire system using a set of simple procedures described in detail in Brennen

(1994). This allows one to proceed to a determination of whether or not a system is stable or unstable given the boundary conditions acting upon it.

The transfer functions for many simple components are readily identified (see Brennen 1994) and are frequently composed of impedances due to fluid friction and inertia (that primarily contribute to the real and imaginary parts of T_{12} respectively) and compliances due to fluid and structural compressibility (that primarily contribute to the imaginary part of T_{21}). More complex components or flows have more complex transfer functions that can often be determined only by experimental measurement. For example, the dynamic response of pumps can be critical to the stability of many internal flow systems (Ohashi 1968, Greitzer 1981) and consequently the transfer functions for pumps have been extensively explored (Fanelli 1972, Anderson *et al.* 1971, Brennen and Acosta 1976). Under stable operating conditions (see sections (Nrc), (Nsi)) and in the absence of phase change, most pumps can be modeled with resistance, compliance and inertance elements and they are therefore dynamically passive. However, the situation can be quite different when phase change occurs. For example, cavitating pumps are now known to have transfer functions that can cause instabilities in the hydraulic system of which they are a part. Note that under cavitating conditions, the instantaneous flow rates at inlet and discharge will be different because of the rate of change of the total volume, V, of cavitation within the pump and this leads to complex transfer functions that are described in more detail in section (Nsi). These characteristics of cavitating pumps give rise to a variety of important instabilities such as cavitation surge (see section (Nri)) or the Pogo instabilities of liquid-propelled rockets (Brennen 1994).

Much less is known about the transfer functions of other devices involving phase change, for example boiler tubes or vertical evaporators. As an example of the transfer function method, in the next section we consider a simple homogeneous multiphase flow.